RELATIVITY AND COSMOLOGY II

Problem Set 5 19th March 2024

1. Dipole anisotropy of the Cosmic Microwave Background

In the lecture we saw that the Universe is filled with the Cosmic Microwave Background (CMB) radiation with the temperature T_{CMB}. Given that the Earth is moving inside this medium with the velocity $\mathbf{v}$, show that the observed temperature of CMB is given by

$$T_{OBS} = T_{CMB}\left(\frac{\sqrt{1-v^2}}{1-v\cos\theta}\right) \, .$$

Here θ is the angle between the direction of observation and Earth's velocity, and $v = |\mathbf{v}|$. Using the above, estimate v for a relative dipole anisotropy $(T_{OBS} - T_{CMB})/T_{CMB}$ of the order of 10^{-3}.

2. Effective number of degrees of freedom

The Friedmann equation in the radiation dominated epoch can be conveniently written in the form

$$H^2 = \frac{8\pi G}{3}\rho, \qquad \rho = g_* \frac{\pi^2}{30} T^4, \tag{1}$$

where ρ is a total energy density of the radiation and g_* is an effective number of degrees of freedom.

1. Show that the value of g_* depends on the total number of degrees of freedom of relativistic bosons g_b and fermions g_f that are present in the plasma in the following way:

$$g_* = g_b + \frac{7}{8} g_f. \tag{2}$$

2. Considering only particles from the Standard Model of particle physics (see table below), calculate the numerical values of g_* for the following temperatures of the plasma in the Early Universe: $T = 1$ TeV, 10 GeV, 10 MeV, 0.1 MeV.

   Hint 1: Assume that at a given temperature T, plasma consists only of particles with $m < T$. (Contributions from particles with $m > T$ are suppressed by a Boltzmann factor $e^{-m/T}$ and can be neglected.)
   Hint 2: Take into account that at the temperature around $T_{\text{QCD}} \sim 200$ MeV QCD phase transition occurs. This means that for $T > T_{\text{QCD}}$ you should consider quarks and gluons as free particles, while for $T < T_{\text{QCD}}$ they can exist only inside baryons because of the confinement.

Particle	Mass (GeV)	Type	DOF
Not strongly interacting particles			
γ	0	boson	2
$\nu_{e,\mu,\tau}$	$\sim 10^{-11}$	fermions	$3 \cdot 2 = 6$
e	$5.11 \cdot 10^{-4}$	fermion	4
μ	0.106	fermion	4
τ	1.78	fermion	4
$W^{\pm}$	80.4	boson	$2 \cdot 3 = 6$
Z^0	91.2	boson	3
h	125	boson	1
Strongly interacting particles			
g	0	bosons	$8 \cdot 2 = 16$
u	$\sim 2 \cdot 10^{-3}$	fermion	$3 \cdot 4 = 12$
d	$\sim 5 \cdot 10^{-3}$	fermion	$3 \cdot 4 = 12$
s	0.095	fermion	$3 \cdot 4 = 12$
c	1.25	fermion	$3 \cdot 4 = 12$
b	4.2	fermion	$3 \cdot 4 = 12$
t	173	fermion	$3 \cdot 4 = 12$

3. Thermodynamics of non-relativistic medium

Consider a gas comprised of non-relativistic particles and antiparticles in equilibrium. Let the mass of the particles be m and their chemical potential μ. Assume that $m/T \gg 1$ and $(m - \mu)/T \gg 1$, where T is the temperature of the gas.

1. Find the number densities of particles and antiparticles.

2. Find the energy density ρ of the gas.

3. Find the pressure p of the gas.
   Hint: You may use either the generic standard definition of pressure as the rate of momentum transfer through the unit surface or the definition of pressure in terms of the stress-energy tensor of an ideal fluid derived in the previous semester.

4. Find the entropy density s of the gas.