RELATIVITY AND COSMOLOGY II

Problem Set 4 12th March 2024

1. Measuring dark energy by luminosity distance

The accelerated expansion of the Universe was discovered by observation of the luminosity distance of supernovae type Ia (SN Ia). However we do not know what is the source of the expansion. The simplest model of the expansion is just a cosmological constant Λ, which can be considered as some form of matter with the equation of state $p = -\rho$. However, it is not excluded that the source of the expansion is a different form of "dark energy" – some exotic matter with the equation of state $p = \omega\rho$, $\omega \approx -1$. Let us try to distinguish between these two scenarios using luminosity distance measurements of SN Ia.

1. Consider a Universe filled with matter with $\Omega_M = 0.3$ and a Λ term with $\Omega_\Lambda = 0.7$. Find an expression for the luminosity distance $d_{L,\Lambda}(z)$ in this case.

2. Now consider a Universe with the same Hubble constant H_0 as in the previous case, but filled with matter with $\Omega_M = 0.3$ and dark energy with $\Omega_{\mathrm{DE}} = 0.7$ that has equation of state $p = -0.9\rho$. Find an expression for the luminosity distance $d_{L,\mathrm{DE}}(z)$ in this case.

3. Good quality data on SN Ia exists up to the redshift $z = 1$. Using previous results make an estimate to what precision one should measure luminosity distance at $z = 1$ for two previous scenarios to distinguish between them?
   Hint: Find the numerical value of $|d_{L,\Lambda}(1) - d_{L,\mathrm{DE}}(1)|/d_{L,\Lambda}(1)$ by numerical integration, e.g., using `Mathematica`.

4. The approximate solution of the previous task may be found using Hubble's law from problem 4 of the previous sheet, where we defined $q_0 = -\frac{1}{H_0^2 R_0}\ddot{R}_0$.
   Start by showing that $q_0 = \frac{1}{2}(\Omega_M - 2\Omega_\Lambda)$ for matter-Λ universe, and $q_0 = \frac{1}{2}(\Omega_M - 1.7\Omega_{DE})$ for dark energy model proposed. Use it to compute $|d_{L,\Lambda}(1) - d_{L,\mathrm{DE}}(1)|/d_{L,\Lambda}(1)$.
   Why are the answers in (rough) agreement despite large z?

2. Fate of the Universe

The purpose of this exercise is to study the different areas and limits in the graph (Ω_M, Ω_Λ) shown below.

1. Identify areas where the universe is open or closed and those where it is accelerating or decelerating.

2. In the case of a universe dominated by the cosmological constant Λ, $p = \rho = 0$, identify areas where there is singularity in the past (the Big Bang).
   Indication: consider the cases $\Lambda > 0$ and $\Lambda < 0$ separately.

3. In the case of a universe dominated by matter, $p = \Lambda = 0$, show that there is an initial singularity. In what area do we have a collapse in the future?

4. Identify areas where there is no initial singularity, and those where the universe expands or collapse in the future.
   Note: At one point of this exercise a cubic equation needs to be solved. Use `Mathematica`.

5. Answer this poll on The true fate of the universe