RELATIVITY AND COSMOLOGY II

Problem Set 3 5th March 2024

1. Evolution of the universe

1. Find the evolution of a flat universe parameterized by the following parameters:
   (a) a flat universe dominated by radiation, $p = \rho/3$, $k = \Lambda = 0$,
   (b) a flat universe dominated by the positive cosmological constant, $\Lambda > 0$, $k = p = \rho = 0$.
2. Consider a universe dominated by a negative cosmological constant, $\Lambda < 0$, $p = \rho = 0$. What can one say about the sign of its curvature? Find the evolution of such a universe.

2. Age of the universe

Find the age of the flat universe composed of non-relativistic matter, at the moment when the Hubble constant $H(t_0) = H_0 \approx 70 \frac{\text{km}}{\text{s Mpc}}$.

3. Stages of evolution

1. Find the scale factor $R(t)$ for the following cases:
   (a) Flat universe composed of radiation and cosmological constant $\Lambda > 0$.
   (b) Flat universe composed of non-relativistic matter and cosmological constant $\Lambda > 0$.
2. For the last case, verify that the age of the universe is given by:

$$t_0 = \frac{2}{3H_0} \frac{1}{\sqrt{1-\Omega_m}} \ln \frac{1+\sqrt{1-\Omega_m}}{\sqrt{\Omega_m}} ,$$

   where $\Omega_m = \rho_m(t_0) \frac{8\pi G}{3H_0^2}$.
3. Show that early-stage evolution of (a) and (b) correspond to formulas found in problem 1.1a and 2, respectively.
4. Show that late-stage evolution of (a) and (b) correspond to solution of problem 1.1b.

4. First correction to the Hubble law

In the lecture, we derived an approximate relation (the Hubble law), that connects the photometric distance d of an object with the redshift z (for $k = 0$ and $\Lambda = 0$),

$$d \simeq H_0^{-1} \cdot z \; ,$$

where $H_0 = \frac{\dot{R}(t_0)}{R(t_0)}$ is the Hubble constant.
This was obtained by approximating $R(t_1) = R(t_0) \approxeq \dot{R}(t_0)(t_1 - t_0)$. However, to get more insight into geometry of universe, more precise relations may be derived.
Again, consider the situation where the difference between time of broadcast and reception of light signal $\Delta t = t_0 - t_1$ is small with respect to H_0^{-1}, but this time try to find the relation between photometric distance d and redshift z up to the order of z^2.
Express your result in Hubble constant H_0 and deceleration parameter q_0, defined as:

$$q_0 \equiv -\frac{\ddot{R}(t_0)}{H_0^2 R(t_0)} \; .$$

Some useful formulae:

$$d = R(t_0)^2 \frac{\bar{r}_1}{R(t_1)} \tag{1}$$

$$\frac{R(t_0)}{R(t_1)} = 1 + z \tag{2}$$

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^{\bar{r}_1} \frac{d\bar{r}}{\sqrt{1 - k\bar{r}^2}} = \bar{r}_1 + \frac{k\bar{r}_1^3}{6} + O(\bar{r}^5). \tag{3}$$

A possible strategy to find these corrections to the Hubble law could be to

- Expand the integrand in (3) around t_0 and integrate do get $\bar{r}_1$ as a power series of $\Delta t = (t_0 - t_1)$. If you are considering only quadratic terms, would the spacetime curvature be included?
- Use the wits of your mathematical analysis to expand $\frac{1}{R(t_1)}$ around t_0 in (1) (2). Plug $\bar{r}_1$ into (1). This should get you d and z in terms of power series in Δt.
- Finally, reduce Δt between two equations. This should get you Hubble law with desired correction.