RELATIVITY AND COSMOLOGY II

Problem Set 2 27th February, 2024

1. Energy-momentum tensor for a perfect fluid

A perfect fluid is a fluid without viscosity and heat conduction. At each point x, the fluid is such that an observer moving with the fluid sees it as isotropic.

1. Argue that in the reference frame where the fluid is locally at rest, its energy-momentum tensor is given by

$$T^{00} = \rho \ , \quad T^{0i} = 0 \ , \quad T^{ij} = p \ \delta^{ij} \ , \tag{1}$$

where ρ is the mass density and p the pressure of the fluid. By boosting to a moving frame, find the expression of $T^{\mu\nu}$ in an arbitrary reference frame. *Indication:* For rewriting Eq. (1) in a covariant way, it is helpful to use the four-velocity u^μ of the fluid as well as the Minkowski metric $\eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

2. Let us now consider a perfect fluid in a flat Minkowski background. Our goal is to explore the implications of the conservation law $\partial^\mu T_{\mu\nu} = 0$ in the non-relativistic limit. (It can be helpful to restore units of c.)

   (a) Before taking the non-relativistic limit, plug the explicit form of $T_{\mu\nu}$ into the equation $u^\nu \partial^\mu T_{\mu\nu} = 0$.

   (b) Show that in the non-relativistic limit, the above result reduces to the continuity equation of fluid mechanics

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \ .$$

   (c) Plug the result from point (a) into the conservation law $\partial^\mu T_{\mu\nu} = 0$ and show that in the non-relativistic limit, the obtained equation reduces to Euler's equation

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \vec{\nabla})\vec{u} + \vec{\nabla} p = 0 \ .$$

   Note: This equation will play an important role in the discussion of gravitational instabilities at the end of the semester.

2. General equation of state

Starting from the conservation of the energy-momentum tensor of a perfect fluid in arbitrary coordinate systems, show that for the flat FLRW metric

$$\frac{d}{dt}(\rho R^3) + p\frac{d}{dt}R^3 = 0. \tag{2}$$

where $R(t) = a(t)R_0$, with R_0 the typical comoving scale of the universe. Now consider the flat Universe with cosmological constant $\lambda = 0$ filled with matter with the following equation of state

$$p = w\rho, \quad w > -1, \quad w = \text{const} \tag{3}$$

1. From Eq.(2) determine the dependence $\rho = \rho(R)$.
2. From the Friedmann equation find $R(t)$ and $\rho(t)$. How do they behave at $t \to 0$?
3. Find which values of w correspond to the accelerating expansion of the Universe (i.e. $\ddot{R} > 0$).

3. Einstein universe

We consider the static Einstein model with a nonzero cosmological constant. Knowing the radius of the universe $R_0 \sim 1.8 \times 10^{10}$ l.y., find the matter density ρ and the cosmological constant λ required in maintaining the universe static. Give answers in units MKSA and GeV.