RELATIVITY AND COSMOLOGY II

Problem Set 1 20th February 2024

1. Homogeneous and isotropic space

Show that the following line elements correspond to homogeneous and isotropic space with constant spatial curvature:

$$ds^2 = R^2 \left(d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \; d\phi^2\right)\right),$$
$$ds^2 = R^2 \left(d\chi^2 + \chi^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2\right)\right),$$
$$ds^2 = R^2 \left(d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta \; d\phi^2\right)\right).$$

2. Volume in curved spacetime

1. Consider the manifold (of constant and positive curvature), as derived in the lecture:

   $$ds^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

   Integrate it to obtain the volume of this space.

2. Consider the (same) manifold with different parametrization, as shown in the previous exercise:

   $$ds^2 = R^2 \Bigg(d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \; d\phi^2\right)\Bigg).$$

   Integrate it to obtain the volume of this space.

3. Discuss the difference of the results.

Note: The volume is given by: $V = \int d^3x \sqrt{\gamma}$, with γ the determinant of the metric.

3. Friedmann–Lemaître–Robertson–Walker (FLRW) metric

A homogeneous and isotropic universe can be described by the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right].$$

Note that $k = \pm 1$ or 0. The above for $k = 0$ can also be written as

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right].$$

For $k = 0$ and $k \neq 0$, perform the following steps:

1. Write $g_{\mu\nu}$ and determine $g^{\mu\nu}$.

2. Derive the geodesic equations for a particle in this space. You may start from the simple action

$$S = m \int d\tau \; g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu.$$

   Here $\dot{x}^\mu = \frac{d}{d\tau}x^\mu$ and τ is the proper time.

3. Deduce by identification the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ by writing the equation of motion as $\ddot{x}^\lambda = -\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. Verify some results with the usual formula:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}\left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}\right), \quad \text{where} \quad \partial_\mu = \frac{\partial}{\partial x^\mu}.$$

4. Calculate the Ricci tensor $R_{\mu\nu}$.

5. Determine the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$.

6. Calculate the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.