RELATIVITY AND COSMOLOGY II

Problem Set 14

1. de Sitter coordinate systems

The d-dimensional de Sitter space dS_d can be realized as the hypersurface described by the equation

$$-X_0^2 + X_1^2 + \cdots + X_d^2 = \ell^2 \tag{1}$$

in flat $d+1$-dimensional Minkowski space.

1. Global coordinates. Write down the metric after the following change of variables:

$$X^0 = \sinh t,$$
$$X^i = \omega^i \cosh t$$

   where ω^i are the spherical coordinates.

2. Poincare patch. Write down the metric after the followng change of variables:

$$X^0 = \sinh t - \frac{1}{2} x_i x^i e^{-t},$$
$$X^i = x^i e^{-t}, \qquad i = 1, \ldots, d-1,$$
$$X^d = \cosh t - \frac{1}{2} x_i x^i e^{-t}$$

   Which part of the dS space doest it describe?

3. Static patch. Write down the metric after the followng change of variables:

$$X^0 = \sqrt{1-r^2} \sinh t$$
$$X^a = r\omega^a, \qquad a = 1, \ldots, d-1,$$
$$X^d = \sqrt{1-r^2} \cosh t$$

   Which part of the dS space doest it describe?

2. Scalar field in FLRW spacetime

Consider a homogeneous scalar field $\phi(t)$ that is described by the action

$$S[\phi] = -\int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right] ,$$

which evolves in a flat FLRW universe with the metric $ds^2 = -dt^2 + a(t)^2 \mathbf{d}\boldsymbol{x}^2$.

1. Compare the energy-momentum tensor of the scalar field with that of a perfect fluid and determine the field energy density ρ and pressure p. Find the equation of state parameter w, such that $p = w\rho$.

2. Determine the condition on the scalar field needed for the accelerated expansion.

3. Derive its equations of motion,

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \ , \\ H^2 = \dfrac{8\pi G}{3}\left(\dfrac{1}{2}\dot{\phi}^2 + V(\phi)\right) \ . \end{cases}$$

3. Behavior of the inflaton

Consider the free massive scalar field in FLRW spacetime, whose dynamics is determined by (see previous exercise)

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \ , \tag{2a}$$

$$H^2 = \frac{8\pi G}{6}\left(\dot{\phi}^2 + m^2\phi^2\right) \ . \tag{2b}$$

1. Show that the above system of equations can be reduced to a first order differential equation for $\dot{\phi}$ that reads

$$\frac{d\dot{\phi}}{d\phi} = -\frac{\dot{\phi}\sqrt{12\pi G}\sqrt{\dot{\phi}^2 + m^2\phi^2} + m^2\phi}{\dot{\phi}} \ ,$$

   where $\dot{\phi}$ is considered as a function of ϕ.

2. Simplify this equation in the following two regimes

   (a) Ultra-hard, i.e. $\dot{\phi} \gg m\phi$ and $\dot{\phi}^2 \gg \frac{m^2}{\sqrt{G}}\phi$.

   (b) Slow-roll, i.e. $d\dot{\phi}/d\phi \approx 0$ and $\dot{\phi}^2 \ll m^2\phi^2$.

   In both cases, solve the resulting equation and determine how the field as well as the Hubble parameter depend on time.

3. Assuming that $\dot{\phi} \sim m\phi$ and $H \ll m$, show that the Hubble parameter corresponds to the one for a matter dominated Universe

$$H = \frac{2}{3t} \ .$$

   Indication: Use the equations of motion in the form of Eq. (2). In order to solve the second one, consider a change of variables of the form $\dot{\phi} = A\sin\theta$ and $\phi = B\cos\theta$, with some appropriate choices of A and B. Moreover, use that after averaging over timescales $t \gg m^{-1}$, oscillatory terms like $\sin(mt)$ or $\cos(mt)$ can be neglected.

4. Estimate the temperature at the end of the slow-roll period, assuming that all the energy is instantaneously transferred to the Standard Model particles.