RELATIVITY AND COSMOLOGY II

Problem Set 13

1. Cold Dark Matter perturbations in radiation domination

In this exercise we would like to study Cold Dark Matter (CDM) density fluctuations within a radiation dominated background, where the Newtonian potential was computed in class to be ($u_s^{rad} = \sqrt{3}$)

$$\Phi(\eta) = -3\Phi_{(i)} \cdot \frac{1}{(u_s^{rad} k\eta)^2} \left[\cos(u_s^{rad} k\eta) - \frac{\sin(u_s^{rad} k\eta)}{u_s^{rad} k\eta} \right].$$

The linearized covariant conservation of the energy-momentum tensor can be written in the form

$$\delta\rho_\lambda' + 3\frac{a'}{a}\left(\delta\rho_\lambda + \delta p_\lambda\right) + \left(\rho_\lambda + p_\lambda\right)\left(\Delta v_\lambda - 3\Phi'\right) = 0, \tag{1}$$

$$\left[\left(\rho_\lambda + p_\lambda\right) v_\lambda\right]' + 4\frac{a'}{a}\left(\rho_\lambda + p_\lambda\right) v_\lambda + \delta p_\lambda + \left(\rho_\lambda + p_\lambda\right)\Phi = 0, \tag{2}$$

where λ labels the components of the cosmic fluid and the dash indicates the derivative with respect to the conformal time.

1. Starting from the above, derive the evolution equations for the CDM density contrast $\delta_{CDM} = \frac{\delta\rho_{CDM}}{\rho_{CDM}}$ in a radiation dominated background. (**Hint**: remember the relation for background densities $\rho' = -3\frac{a'}{a}(\rho + p)$, and that $u_s^{CDM} = 0$).
2. Deep inside the sound horizon the potential is well-approximated by $\Phi \equiv 0$. Write the general solution to the equations you found at the previous point, in this case. Which one is the decaying mode at early times?
3. Now take into account the actual form of Φ. Solve the equation for v_{CDM} found at point 1 (you can leave it in the form of an indefinite integral). Determine the integration constant by requiring the correct early times behaviour.
4. Finally, solve for the leading behaviour of δ_{CDM} at late time. (Here you can actually compute the integral).

2. Sachs-Wolfe effect

In this exercise, we would like to investigate the effects of the gravitational potential on the CMB temperature fluctuations.

1. To start with, write down the geodesic equation a free photon will follow, starting from the metric written in the conformal Newtonian gauge.
2. Relate the coordinate-frame momentum to the momentum measured in a local inertial frame by an observer. What is (at first order) the equation that governs the evolution of this observed energy? Can you give an interpretation to all three contributions?

3. Assuming instantaneous recombination, compute the temperature fluctuations observed today, separating the contribution of fluctuations at the time of recombination, from the effects due to the gravitational potential.

4. Neglecting the effects given by the velocity of the electrons at recombination, compute the temperature fluctuations at the time of recombination, recalling that it occurred during the radiation-dominated epoch.

3. Horizon problem

The Horizon problem is one of the biggest phenomenological puzzle which commist us to the theory of inflation. In this exercise we want to familiarize a bit with it, and see how inflation elegantly provides a solution.

1. Assuming that the radiation domination was the only stage of the evolution of the Universe, find the angle θ_h at which the particle horizon at the moment of recombination ($t_{\rm rec} = 3.7 \times 10^5$ years) would be observed today ($t_0 = 13.7 \times 10^9$ years).

Now, in order to solve the problem, let's imagine to glue an inflationary phase just before the radiation domination period. Assume that during inflation the energy density is almost constant an equal to $\rho \approx v^4$. After the period of inflation, the Universe is reheated up to the temperature of $T \approx v$.

2. How many e-folds $N = \log(a_e/a_i)$ are needed to solve the horizon problem? Here a_i and a_e are the scale factors at the beginning and at the end of inflation stage.

3. Find the values of N for $v = 10^8$ GeV and $v = 10^{16}$ GeV.