RELATIVITY AND COSMOLOGY II

Problem Set 12 14th May 2024

1. Linearised Einstein tensor for scalar perturbation

Starting with a metric $g_{\mu\nu} = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu})$ derive the components of the Einstein tensor δG^μ_ν to linear order in $h_{\mu\nu}$ in the Newtonian gauge

$$h_{00} = 2\Phi \qquad h_{0i} = 0, \qquad h_{ij} = -2\Psi\,\delta_{ij}.$$

For the components of the Einstein tensor you should get:

$$\delta G^0_0 = \frac{2}{a^2}\left(-\Delta\Psi + 3\frac{a'}{a}\Psi' - 3\frac{a'^2}{a^2}\Phi\right), \tag{1}$$

$$\delta G^0_i = \frac{2}{a^2}\left(-\partial_i\Psi' + \frac{a'}{a}\partial_i\Phi\right), \tag{2}$$

$$\delta G^i_j = \frac{1}{a^2}\partial^i\partial_j\left(\Phi + \Psi\right) - \frac{2}{a^2}\delta^i_j\left[-\Psi'' + \frac{1}{2}\Delta\left(\Phi+\Psi\right) + \frac{a'}{a}\left(\Phi' - 2\Psi'\right) + \left(2\frac{a''}{a} - \frac{a'^2}{a^2}\right)\Phi\right], \tag{3}$$

where $\Delta = \partial_i\partial_i$.

Hint: Note that $g_{\mu\nu}$ and $\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ are related by conformal transformation. Thus the Einstein tensor associated to $g_{\mu\nu}$ is related to the one generated by $\gamma_{\mu\nu}$ as

$$G_{\mu\nu}(g) = G_{\mu\nu}(\gamma) - 2\frac{\nabla_\mu\nabla_\nu a}{a} + 4\frac{\partial_\mu a\partial_\nu a}{a^2} + \gamma_{\mu\nu}\gamma^{\lambda\rho}\left(2\frac{\nabla_\lambda\nabla_\rho a}{a} - \frac{\partial_\lambda a\partial_\rho a}{a^2}\right),$$

where the covariant derivatives are now evaluated with the metric $\gamma_{\mu\nu}$.

2. Linearised energy-momentum conservation

Using the same metric as in the previous problem, consider scalar perturbations of the stress energy tensor for a perfect fluid

$$p \to p + \delta p \qquad \rho \to \rho + \delta\rho \qquad u^0 \to \frac{1}{a}(1 + \delta u^0) \qquad u^i \to \frac{1}{a}(\partial_i v).$$

Find the constraints on δu^0 due to the 4-velocity normalisation.
Write down the linearised Einstein's equations, and show that these imply $\Psi = -\Phi$.
Finally, derive the linearised equations of covariant energy-momentum conservation

$$\delta(\nabla_\mu T^\mu_\nu) = 0,$$

(Recall that you should use the covariant derivative with respect to the metric $g_{\mu\nu}$).
You should find

$$\delta\rho' + 3\frac{a'}{a}\left(\delta\rho + \delta p\right) + (\rho + p)\left(\triangle v - 3\Phi'\right) = 0, \tag{4}$$

$$[(\rho+p)v]' + 4\frac{a'}{a}(\rho+p)v + \delta p + (\rho+p)\Phi = 0. \tag{5}$$

3. Helicity basis tensors

Consider two orthogonal unit polarization vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. How do they transform under rotations in the transverse plane? How to make the vectors with helicities ± 1? Construct the two following tensors

$$
\begin{aligned}
e_{ij}^{(+)} &= \frac{1}{\sqrt{2}}\left(e_i^{(1)}e_j^{(1)} - e_i^{(2)}e_j^{(2)}\right) \\
e_{ij}^{(\times)} &= \frac{1}{\sqrt{2}}\left(e_i^{(1)}e_j^{(2)} + e_i^{(2)}e_j^{(1)}\right)
\end{aligned}
$$

Check that these tensors are traceless and symmetric. Prove that you can construct two linear combinations out of these tensors with helicities ± 2. Moreover, show that any arbitrary symmetric transverse traceless tensor of second rank is a combination of e_{ij}^{+} and $e_{ij}^{\times}$, i.e. a misture of helicities $+2$ and -2.

4. Dimension of helicity basis

Show that the third combination of the unit vectors $\left(e_i^{(1)}e_j^{(1)} + e_i^{(2)}e_j^{(2)}\right)$ from the problem above has zero helicity, i.e. that is transforms trivially under rotations around $\vec{k}$-axis. Express this combination in terms of δ_{ij} and $k_i k_j$.