RELATIVITY AND COSMOLOGY II

Problem Set 11 7th May 2024

1. Jeans instabilities

The system of equations describing the motion of a perfect fluid in the Newtonian framework is

$$\begin{aligned}
&\nabla^2\phi = 4\pi G\rho \ , \\
&\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0 \ , \\
&\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\phi \ .
\end{aligned} \tag{1}$$

1. Consider perturbations on top of the static homogeneous solution $\rho_0 = \text{const.}, p_0 = \text{const.}, \vec{v_0} = 0 \ , \vec{\nabla}\phi_0 = 0$, viz.

   $$\begin{aligned}
   \rho &= \rho_0 + \delta\rho \ , \qquad & p &= p_0 + \delta p \ , \\
   \vec{v} &= \delta\vec{v} \ , & \phi &= \phi_0 + \delta\phi \ .
   \end{aligned}$$

   Determine the system of equations the perturbations satisfy and show that it can be written as

   $$\frac{\partial^2\delta\rho}{\partial t^2} - v_s^2\nabla^2\delta\rho = 4\pi G\rho_0\delta\rho \tag{2}$$

   where $v_s^2 = \delta p/\delta\rho$ is the sound speed.
   Derive the dispersion relation obeyed by plane waves solutions of the equation above, and find the Jeans' wavevector k_J for which the perturbations become unstable. Interpret this result physically.

2. When we take into account the expansion of the Universe, the simplest solution to eqs. (1) is

   $$\rho_0 \propto a^{-3} \ , \ \vec{v_0} = H\vec{r} \ , \ \phi_0 = \frac{2\pi G\rho_0 r^2}{3} \tag{3}$$

   where the evolution of the scale factor is determined by the second Friedmann equation. Show that the perturbations in this case satisfy

   $$\begin{aligned}
   &\nabla^2\delta\phi = 4\pi G\delta\rho \ , \\
   &\frac{\partial\delta\rho}{\partial t} + 3H\delta\rho + H(\vec{r}\cdot\vec{\nabla})\delta\rho + \rho_0\vec{\nabla}\delta\vec{v} = 0 \ , \\
   &\frac{\partial\delta\vec{v}}{\partial t} + H\delta\vec{v} + H(\vec{r}\cdot\vec{\nabla})\delta\vec{v} + \frac{v_s^2}{\rho_0}\vec{\nabla}\delta\rho + \vec{\nabla}\delta\phi = 0 \ .
   \end{aligned}$$

2. Linear sizes of perturbations and masses of objects

When the density perturbation of the present linear size R goes non-linear and collapses, a compact object is formed. Estimate the total mass of such object if its size equals

1. $R \sim (1 - 3)$ Mpc (galaxies)
2. $R \sim (10 - 30)$ Mpc (clusters of galaxies)
3. $R \sim (40 - 400)$ kpc (dwarf galaxies)
4. $R \sim 10$ kpc (halos of protostars)

3. Free streaming length

Consider a particle that decouples in the Early Universe being relativistic, with $p \gg m$, and then propagates freely. Estimate the physical distance that this particle will cover up to the redshift $\mathrm{z} = 3000$ (matter-radiation equality) taking into the account that particle may become non-relativistic during its propagation. This distance is called the free streaming length. Find the mass inside the sphere with the radius of free streaming scale, for m of the order of neutrino mass, $\sim 0.1\,\mathrm{eV}$. This is the minimal mass of the structure which can be formed in the Universe where neutrino plays a role of dark matter.

For simplicity approximate the velocity integral by splitting it into non-relativistic and ultra-relativistic part.