RELATIVITY AND COSMOLOGY II

Problems for Exam Practice – year 2023 10th June 2023

Problem 1

Estimate the current entropy density associated with ordinary matter (relativistic and non-relativistic) of the Universe. Consider $T_\gamma = 2.75$ K, assume that the neutrinos are massless and that the baryons are distributed homogeneously.

Problem 2

At $T = 1\,\text{GeV}$ cosmic plasma consists of the leptons (electrons, muons, neutrinos and their antiparticles), photons and three light quarks ($m_u \approx 2\,\text{MeV}$, $m_d \approx 5\,\text{MeV}$, and $m_s \approx 100\,\text{MeV}$). Note also that each quark can be in three color states. The quark's baryon number equals 1/3.
Find the chemical potential of the u-quark at $T = 1\,\text{GeV}$. Assume that the baryon-to-entropy ratio is $\frac{n_B - n_{\overline{B}}}{s} \sim 10^{-9}$ at that moment and that $\mu_u = \mu_d = \mu_s$.

Problem 3

Estimate the number density of antibaryons at the moment of freezing out the reactions of baryon-antibaryon annihilation ($T_f \approx 10$ keV).
Indication: Neglect the mass difference between proton and neutron. Recall that in equilibrium $n_B^{eq} = n_\gamma^{eq}\eta_B$, $\eta_B \sim 10^{-9}$.

Problem 4

If there exist neutrinos of ultra-high energies in Nature, they may scatter off relic neutrinos. The neutrino-neutrino cross section in the Standard Model is very small. Its maximum value, $\sigma = 0.15\ \mu b = 1.5 \times 10^{-31}\,\text{cm}^2$, is reached at center-of-mass energy $E_0 \simeq M_Z \approx 90\,\text{GeV}$. Find the mean free path of such neutrino in the present Universe and its energy in the CMB frame.

Problem 5

Find the size of horizon l_H at the moment of transition from quark-gluon plasma to hadron states that occurs at $T \approx 150\,\text{MeV}$. Find the value of this size at the moment of matter-radiation equality ($T_{eq} \approx 1\,\text{eV}$).

Problem 6

Consider the hypothetical universe filled with baryons, photons and neutrinos, that contains no asymmetry between baryons and antibaryons, $\eta_B = 0$. Being in equilibrium, the baryons participate in reactions of production and annihilation whose velocity averaged cross section in non-relativistic regime is approximately $\langle \sigma v \rangle \simeq \sigma_0 \approx 100\,\mathrm{GeV}^{-2}$. Assume that these reactions freeze out at $T_f \approx 20\,\mathrm{MeV}$. Find the present value of the baryon-to-photon ratio n_B/n_γ in this universe.

Problem 7

Assume that dark matter is made of neutral bosons that went out of equilibrium at $T = 0.23\,\mathrm{eV}$. What must the mass of these particles be in order to account for the present abundance of dark matter? How many particles do we expect to find in average in one cubic centimeter?
Indication: Assume that the particles decoupled being nonrelativistic.

Problem 8

At high temperatures, a fermion ψ with mass $m_\psi = 1\,\mathrm{GeV}$ is maintained in thermal equilibrium by the reaction $\overline{\psi}\psi \leftrightarrow \gamma\gamma$, that decouples at $T = 50\,\mathrm{MeV}$. Find the velocity averaged cross section of this reaction $\langle \sigma v \rangle$, assuming zero chemical potential for ψ.

Problem 9

Are the neutrinos at the time of photon decoupling relativistic? What is the shape of their number density distribution $n(p)$?
Give the answer assuming that the total mass of neutrinos ($\sum_\nu m_\nu$) is 0.06 eV, and the difference of the squares of the masses are:

$$m_2^2 - m_1^2 = 8 \times 10^{-5}\,\mathrm{eV}^2, \qquad m_3^2 - m_2^2 = 2 \times 10^{-3}\,\mathrm{eV}^2.$$

Problem 10

Interaction of proton with photon at sufficiently high energies may lead to the absorption of photon and creation of π-meson: $p + \gamma \to p + \pi^0$. Let the cross section of the latter process in the center-of-mass frame be

$$\sigma = \begin{cases} 0, & \text{at } E < E_0, \\ 5 \times 10^{-28}\,\mathrm{cm}^2, & \text{at } E > E_0, \end{cases}$$

where E is the total energy of photon and proton in the center-of-mass frame, $E_0 =$ 1200 MeV. Find the mean free path of a proton in the present Universe with respect to this process as a function of proton energy in the CMB frame.
Hint: Ignore all photons (e.g., emitted by stars), except for CMB.

Problem 11

What is the Jeans length at the time of neutrino decoupling?
Hint: The temperature of neutrino decoupling is $T_d = 2\,\mathrm{MeV}$.

Problem 12

For bosonic dark matter particles to form the clusters in galaxies, their de Broglie length must be smaller than the galaxy size. Given that the sizes of dwarf galaxies are of order $l \sim 1\,\text{kpc}$, and velocities of dark matter particles are of order $v \sim 10^{-5}\,c$, put the lower bound on the mass of dark matter bosons.

Problem 13

Find the evolution of a universe dominated by "strong" matter, i.e. $\rho = p$, $\Lambda = k = 0$.

Problem 14

Can we have a static universe with $k \neq 0,\ \rho \neq 0,\ p \neq 0$ and $\Lambda = 0$? What type of matter (equation of state) is needed?

Problem 15

Consider the Universe filled with matter whose equation of state is that of Chaplygin gas,

$$p = -\frac{A}{\rho}, \quad A > 0.$$

1. Find the dependence of the energy density of this matter on scale factor R.
2. Find the behavior of the scale factor R on time in the limit of small and large R. Compare this behavior with that of ordinary matter ($p = 0$) and dark energy ($p = -\rho$).

Problem 16

Consider a spatially flat universe filled with nonrelativistic matter and dark energy whose equation of state is $p = w\rho$. Suppose that at present time $t = t_0$, the abundances of matter and dark energy are Ω_m and Ω_{DE} respectively, while the Hubble parameter is equal to H_0. Using this information, derive the expression for the age of such Universe t_0 (in the form of integral). In which case the age of the Universe is larger, for $w = 0.9$ or for $w = 1.1$?
Hint: First, find the dependence of ρ_{DE} on scale factor R.

Problem 17

Consider the closed universe filled with nonrelativistic matter. Find the time between the initial singularity and a final collapse (i.e. the total lifetime of the universe). Express it through the total mass of the universe.
Hint: Find the dependences of the scale factor R and time t on the conformal time η.

Problem 18

Consider the Universe filled with cold matter ($\Omega_m = 0.3$), and dark energy ($\Omega_{DE} = 0.7$)

at the time t_0. Assume that the dark energy component has an equation of state of the form $p = w\rho$, where $-1 < w < -\dfrac{1}{3}$. What is the dependence of energy density for dark energy ρ on a scale factor R? Find the value of redshift $z = z(w)$, counted from t_0, as a function of w, at which the transition from deceleration to acceleration occurs. For which value of w this transition would occur at $t = t_0$?

Problem 19

The transition between two atomic states leads to emission of a photon. Knowing that such a transition is observed in the laboratory with energy release $E_0 = 10\,\text{keV}$ and that the same transition is observed in a galaxy by detecting the photon with energy $E = 1\,\text{keV}$, deduce what was the age of the Universe when the emission took place. Consider that the Universe is always matter dominated.

Problem 20

When the electron-positron annihilation freezes out at $T \simeq m_e/40$, the relation between the number of electrons e^- and the number of positrons e^+ is $\frac{n_{e^-} - n_{e^+}}{n_{e^-} + n_{e^+}} \approx 10^{-10}$. What are the chemical potentials of these particles at the that time?

Problem 21

Assume that the universe contains only photons γ and a cosmological constant Λ. Let the exponential expansion takes over the radiation domination epoch at $T = 10^{12}\,\text{GeV}$. Estimate the value of Λ and the value of Hubble's parameter at that moment.

Problem 22

Suppose that the densities of photons, protons and electrons have the following constant ratios: $n_p = n_e = 10^{-9} n_\gamma$ at the temperatures $T < m_e$. Find the temperature at which the energy density of nonrelativistic matter equals the energy density of radiation.

Problem 23

Are the neutrinos at the time of photon decoupling relativistic? What was their momentum at that time?
Give the answer assuming that the total mass of neutrinos ($\sum_\nu m_\nu$) is 3 eV, and the difference of the squares of the masses are:

$$m_2^2 - m_1^2 = 8 \times 10^{-5}\,\text{eV}^2, \qquad m_3^2 - m_2^2 = 2 \times 10^{-3}\,\text{eV}^2.$$

Problem 24

Estimate the average speed of scalar particles with mass $m = 1\,\text{GeV}$ at the time of matter-radiation equality ($T_{eq} = 0.7\,\text{eV}$) assuming that they were in thermal equilibrium until 50 keV.

Problem 25

Assume that neutrinos are massless. What is the redshift of neutrinos emitted at their decoupling in Early Universe and registered today?

Problem 26

In this exercise we would like to study adiabatic density fluctuations of the baryon-photon component at the time from radiation-matter equality η_{eq} up to recombination η_{rec}. The Newtonian potential was approximately time-independent for modes with $u_s k \eta_{eq} \gg 1$, and it is given by the expression

$$\Phi = \frac{81}{4}\Phi_{(i)} \cdot \frac{H_0^2 a_0^2}{k^2}\Omega_{CDM}(1+z_{eq}) \log\left(0.2k\eta_{eq}\right). \tag{1}$$

The linearized covariant conservation of the energy-momentum tensor can be written in the form

$$\delta\rho_\lambda' + 3\frac{a'}{a}\left(\delta\rho_\lambda + \delta p_\lambda\right) + \left(\rho_\lambda + p_\lambda\right)\left(\nabla^2 v_\lambda - 3\Phi'\right) = 0, \tag{2}$$

$$\left[\left(\rho_\lambda + p_\lambda\right)v_\lambda\right]' + 4\frac{a'}{a}\left(\rho_\lambda + p_\lambda\right)v_\lambda + \delta p_\lambda + \left(\rho_\lambda + p_\lambda\right)\Phi = 0, \tag{3}$$

where λ labels the components of the cosmic fluid and the dash indicates the derivative with respect to the conformal time.

For $\eta_{eq} \leq \eta \leq \eta_{rec}$ the baryon-photon plasma is a single medium, due to their tight coupling. This means that the physical velocities of photons and baryons are the same, $v_\gamma = v_B \equiv v_{B\gamma}$, and that equation (3) only applies to the global densities of the photon-baryon fluid.

Nevertheless, baryon density is independently conserved, hence equation (2) also holds separately for the photon and baryon components.

1. Show that for adiabatic modes it holds $\delta_B = \frac{3}{4}\delta_\gamma$, where $\delta_\lambda = \frac{\delta\rho_\lambda}{\rho_\lambda}$ is the density contrast of the $\lambda-$species.

2. Define the baryon-photon ratio to be $R_B := \frac{3\rho_B}{4\rho_\gamma}$. Compute the speed of sound of the baryon-photon plasma in terms of R_B.

3. Starting from equations (2) and (3), derive the evolution equations for the photon density contrast δ_γ in a matter dominated background.

   Hint: recall the relation for background densities $\rho_\lambda' = -3\frac{a'}{a}(\rho_\lambda + p_\lambda)$.

The equation derived at the previous point is a second order ordinary differential equation in δ_γ, with a source term depending on Φ, the gravitational potential.

The general solution to the homogeneous part of the equation can be derived with the WKB approximation and is of the form

$$\delta_\gamma^{hom}(\eta) = A\left(3u_s^2\right)^{\frac{1}{4}} \cos\left(k\int_{\eta_\star}^{\eta} \mathrm{d}\tilde{\eta} u_s\right), \tag{4}$$

where A and $\eta_\star$ are integration constants.

4. Write down the full expression for photon perturbations in the tight-coupling approximation, during matter domination. Remember that during radiation domination acoustic oscillations were derived in class to be $\delta_\gamma^{rad}(\eta) = 6\Phi_i \cos(u_s k \eta)$.

Problem 27

In this exercise you will conduct a detailed analysis of several causal structures in an expanding universe, carrying out some explicit computations in the simplified case of a matter dominated FLRW universe.

1. Start by recalling the definition of critical density and the relative energy density contributions Ω_i. Derive the following equation

$$\frac{H^2}{H_0^2} = \Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_K \left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda.$$

From now on, always assume to be in a FLRW matter dominated universe ($\Omega_\Lambda = \Omega_R = 0$, $\Omega_M = 1$, $\Omega_K = 0$), and unless otherwise stated the required distances can be expressed in *comoving* coordinates.

2. Compute the proper (physical) and the comoving distances from an observer to a galaxy seen at redshift $z_\star$ by an observer at the origin at time t_0. What is the present proper size of the observable universe, in terms of its age t_0?

3. The *particle horizon* at time $\tilde{t}$ is defined as the maximum distance light might have travelled from $t = 0$ to $t = \tilde{t}$. Compute this as well.

4. The *effective* speed is defined as the time derivative of the proper distance seen by the observer, $v_{eff}(\tilde{t}) := \frac{\mathrm{d}}{\mathrm{d}t} d_P(t)\Big|_{\tilde{t}}$. Beyond which distance is the effective speed of a photon such that it moves away from the observer? This quantity defines the Hubble sphere $D_H(\tilde{t})$ at time $\tilde{t}$.

5. In a spacetime diagram with conformal time on the vertical axis and comoving distance on the horizontal axis, draw the following quantities (computed for the matter dominated universe):

   ⋆ Past light-cone of an observer sitting at the origin at present time;

   ⋆ World-line of a galaxy at comoving distance ℓ;

   ⋆ Particle horizon (centered at the origin);

   ⋆ Hubble sphere (centered at the origin). Is this surface a causal horizon?

Problem 28

Imagine there existed a neutral spin $\frac{3}{2}$ particle χ, of mass $m_\chi = 10\,\mathrm{keV}$, which interacts with the standard model sector only through the reaction

$$\chi\chi \leftrightarrow e^+e^-$$

1. Find the freeze out temperature of this particle assuming thermal equilibrium and that the cross section is $\sigma = \frac{\alpha}{m_\chi^2}$, with $\alpha \sim \mathcal{O}(10^{-27})$ (The chemical potential can be taken to be negligible).

2. After χ freeze out only neutrinos, electrons, photons and baryons are left interacting in the plasma. Use the conservation of entropy to estimate the ratio between the effective temperature of the χ particles and the temperature of photons, after e^+e^- freeze-out.

3. Determine the present cosmological abundance of χ particle: is the presence of these particles consistent with cosmological observations? For this recall $T_{\gamma,0} \sim 2.34 \times 10^{-13}$GeV and $\rho_{crit} \sim 3.7 \times 10^{-47}$GeV^4.

Problem 29

1. Recall the definition of a deceleration parameter $q = -\frac{\ddot{a}}{aH^2}$. Using the Friedmann equations find a general expression for q in terms of Ω_i. Show that in the Universe which contains only matter and $\Lambda > 0$:

$$q = \frac{1}{2}\Omega_m - \Omega_\Lambda \tag{5}$$

2. Solve Friedmann equations in the flat Universe with matter and cosmological constant. Define the time origin by $a(0) = 0$. Express your result in the form

$$a(t) = A \cdot \sinh^{2/3}\left(\frac{3H_0}{2}t\sqrt{1 - \Omega_{m,0}}\right) \tag{6}$$

   where H_0 is the Hubble expansion rate today and $\Omega_{m,0}$ is the matter density parameter today.
   Hint: Use the change of variables $a = x^\alpha$.

3. Find the age of the universe, t_0, in terms of $\Omega_{m,0}$. Use the normalization $a_0 \equiv a(t_0) = 1$ to determine the integration constant A in formula (6).
   Hint: Use the formula

$$\text{arccoth}\, x = \frac{1}{2}\log\frac{x+1}{x-1}. \tag{7}$$

4. If today in the Universe $\Omega_{m,0} = 0.3$, calculate the redshift, z, at the moment when the expansion changes from the deceleration to acceleration. Also calculate the present-day value of the deceleration parameter, q_0.

Problem 30

Consider massive particles and antiparticles with mass m and number densities $N(t)$ and $\bar{N}(t)$. If they interact with cross-section σ at velocity v, the evolution of $N(t)$ is described by

$$\frac{\partial N}{\partial t} + 3HN = -(N\bar{N} - N_{eq}\bar{N}_{eq})\langle\sigma v\rangle. \tag{8}$$

The first term to the right represents the rate of annihilation of particles with anti-particles, while the second one represents the particle production rate, expressed in terms of the equilibrium densities N_{eq} and $\bar{N}_{eq}$. This equation is valid even for large deviations of N from N_{eq}.

1. Considering the analogous equation for the antiparticles, show that

$$(N - \bar{N})a^3 = \text{const.}$$

Now consider a radiation-dominated universe, where the numbers of particles and antiparticles are initially the same.

You may find it useful to choose a reference scale such that $a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}} = \frac{m}{T}$, where T is the radiation temperature.

2. Defining $w \equiv \frac{N}{T^3}$, show that

$$\frac{dw}{da} = -\frac{\lambda}{a^2}(w^2 - w_{eq}^2), \tag{9}$$

   where $\lambda \equiv \frac{m^3 \langle \sigma v \rangle}{H_\star}$ and $H_\star$ is the Hubble constant when $T = m$.
   Hint: Start by rewriting the left hand side of equation 8 as $\frac{1}{a^3}\frac{\partial (N a^3)}{\partial t}$.

3. If λ is a constant, show that at times much later than freeze out, w approaches the relic value

$$w_\infty = \frac{m}{\lambda T_f},$$

   where T_f is the freeze-out temperature.
   Hint: argue why at late times $w \gg w_{eq}$.

Now apply this to proton-antiproton annihilation, assuming initial symmetry. You may use the following: $\langle \sigma v \rangle \approx 100\ \text{GeV}^{-2}$, at freeze-out the temperature is $T_f \approx 20$ MeV and $g_\star = 10.25$ (since muons and pions have decoupled) and $M_{pl} = 1.2 \times 10^{19}$ GeV.

4. Show that the (anti-)proton to photon ratio is

$$\frac{N}{N_\gamma} = \frac{\bar{N}}{N_\gamma} \approx 10^{-18}$$

   How does this compare with observational data? Which of the assumptions we made should be revised?