RELATIVITY AND COSMOLOGY II

Solutions to the final exam 23rd June 2025

Problem 1

(a) Determine the scaling of the energy density ρ with scale factor a for a component with constant equation of state parameter w. Hence show that the Hubble parameter can be written as $H(z) = H_0 E(z)$, with

$$E(z) = \left[\sum_i \Omega_{i,0} \,(1+z)^{3(1+w_i)} \right]^{1/2}, \tag{1}$$

where you should define $\Omega_{i,0}$ and where the sum is over components i with constant equation of state parameters w_i.

(b) For the rest of the problem, consider a universe containing only cold dark matter (m) and a non-standard dark energy component (DE); the non-standard dark energy component has an equation of state parameter that is not constant, but instead depends on redshift in the following way:

$$w(z) = w_0 + \frac{w_a\, z}{1+z}, \tag{2}$$

where w_0 and w_a are constant parameters. Carefully show that in such a universe the function $E(z)$ is given by

$$E(z) = \Big[\Omega_{m,0}(1+z)^3 + X(\Omega_{DE,0}, z, w_0, w_a) \Big]^{1/2}, \tag{3}$$

where $X(\Omega_{DE,0}, z, w_0, w_a)$ is a function you should specify.

(c) A bright source in a distant galaxy emits photons at a time t_s that are received on earth at time t_0. The source's redshift z_1 is initially measured by astronomers at time t_0. The redshift of the same source is measured a second time, giving z_2, after waiting an interval Δt_0 (which corresponds to an interval Δt_s in the rest frame of the source). Show that the difference $\Delta z \equiv z_2 - z_1$ (referred to as the redshift drift) between the redshifts of the source at times $t_0 + \Delta t_0$ and t_0 is given by

$$\Delta z \;=\; \frac{a(t_0 + \Delta t_0)}{a(t_s + \Delta t_s)} \;-\; \frac{a(t_0)}{a(t_s)}. \tag{4}$$

After relating Δt_s to Δt_0, show that the redshift drift Δz is given by

$$\Delta z = f\big(z_1, \Delta t_0, E(z_1), H_0\big), \tag{5}$$

where you should determine the function $f(z_1, \Delta t_0, E(z_1), H_0)$. You may perform all calculations to linear order in $H\Delta t \ll 1$ and $\Delta t/t \ll 1$.

(d) Assume $w_0 = -1$ and $|w_a| \ll 1$. Given an (exceptionally high) redshift measurement precision of $\delta z \sim 10^{-11}$, determine the smallest value of $|w_a|$ for which the predicted Δz at $z_1 \approx 1$ over $\Delta t_0 = 10$yr would be measurably different from the $w_a = 0$ case. You can approximate $H_0 \approx 10^{-10}\,\mathrm{yr}^{-1}$.

Solution

(a) A fluid with constant equation-of-state parameter w obeys the continuity equation

$$\dot\rho + 3H(1+w)\rho = 0, \quad H = \frac{\dot a}{a}.$$

Hence

$$\frac{d\rho}{\rho} = -3(1+w)\,\frac{da}{a} \implies \rho(a) = \rho_0\, a^{-3(1+w)}. \tag{6}$$

Since the redshift z is related to the scale factor by

$$1+z = \frac{1}{a}, \qquad a = \frac{1}{1+z}, \tag{7}$$

this may be written

$$\rho(z) = \rho_0\,(1+z)^{3(1+w)}. \tag{8}$$

Defining the critical density $\rho_{c,0} = 3H_0^2/(8\pi G)$ and $\Omega_{i,0} = \rho_{i,0}/\rho_{c,0}$, the Friedmann equation $H^2 = (8\pi G/3)\sum_i \rho_i$ gives

$$\frac{H^2(z)}{H_0^2} = \sum_i \Omega_{i,0}\,(1+z)^{3(1+w_i)}, \quad H(z) = H_0\Big[\sum_i \Omega_{i,0}(1+z)^{3(1+w_i)}\Big]^{1/2}. \tag{9}$$

(b) Now consider cold matter ($w_m = 0$) plus dark energy (DE) with

$$w(z) = w_0 + w_a\,\frac{z}{1+z}.$$

The DE density evolves according to

$$\frac{d\rho_{\rm DE}}{\rho_{\rm DE}} = -3\big[1+w(z)\big]\frac{da}{a} = +3\big[1+w(z)\big]\frac{dz}{1+z},$$

hence

$$\rho_{\rm DE}(z) = \rho_{\rm DE,0}\,\exp\!\Big[3\int_0^z \frac{1+w(z')}{1+z'}\,dz'\Big]. \tag{10}$$

We split the integrand:

$$\int_0^z \frac{1+w(z')}{1+z'}\,dz' = \int_0^z \frac{1+w_0}{1+z'}\,dz' + \int_0^z \frac{w_a\,z'}{(1+z')^2}\,dz'. \tag{11}$$

The first term is

$$(1+w_0)\ln(1+z),$$

and for the second we set $u = 1+z'$:

$$\int_0^z \frac{z'}{(1+z')^2}\,dz' = \int_{u=1}^{1+z} \frac{u-1}{u^2}\,du = \big[\ln u + \tfrac{1}{u}\big]_1^{1+z} = \ln(1+z) - \frac{z}{1+z}.$$

Thus

$$\begin{aligned}\int_0^z \frac{1+w(z')}{1+z'}\,dz' &= (1+w_0)\ln(1+z) + w_a\Big[\ln(1+z) - \frac{z}{1+z}\Big] \\ &= (1+w_0+w_a)\ln(1+z) - w_a\frac{z}{1+z}.\end{aligned} \tag{12}$$

Substituting back,

$$\rho_{\rm DE}(z) = \rho_{\rm DE,0}\,(1+z)^{3(1+w_0+w_a)}\exp\!\Big[-3w_a\,\frac{z}{1+z}\Big]. \tag{13}$$

The Friedmann equation then reads

$$\frac{H^2(z)}{H_0^2} = \Omega_{m,0}(1+z)^3 + \Omega_{\rm DE,0}(1+z)^{3(1+w_0+w_a)}\exp\!\Big[-3w_a\,\tfrac{z}{1+z}\Big], \tag{14}$$

or, defining

$$X(\Omega_{\rm DE,0}, z; w_0, w_a) = \Omega_{\rm DE,0}(1+z)^{3(1+w_0+w_a)}\exp\!\Big[-3w_a\,\tfrac{z}{1+z}\Big], \tag{15}$$

$$H(z) = H_0\Big[\Omega_{m,0}(1+z)^3 + X(\Omega_{\rm DE,0}, z; w_0, w_a)\Big]^{1/2}. \tag{16}$$

(c) A source which emitted at time t_s and is observed at t_0 has

$$1+z_1 \;=\; \frac{a(t_0)}{a(t_s)}.$$

After a small interval Δt_0 on Earth and Δt_s at the source,

$$1+z_2 = \frac{a(t_0+\Delta t_0)}{a(t_s+\Delta t_s)} \implies \Delta z \equiv z_2 - z_1 = \frac{a(t_0+\Delta t_0)}{a(t_s+\Delta t_s)} - \frac{a(t_0)}{a(t_s)}. \tag{4}$$

Photons satisfy

$$\int_{t_s}^{t_s+\Delta t_s}\frac{dt}{a(t)} = \int_{t_0}^{t_0+\Delta t_0}\frac{dt}{a(t)} \implies \frac{\Delta t_s}{a(t_s)} = \frac{\Delta t_0}{a(t_0)}.$$

Since $1+z_1 = a(t_0)/a(t_s)$, this gives

$$\Delta t_s = \frac{a(t_s)}{a(t_0)}\,\Delta t_0 = \frac{\Delta t_0}{1+z_1}. \tag{17}$$

Expanding $a(t+\Delta t) \approx a(t)[1+H(t)\,\Delta t]$ to first order,

$$a(t_0+\Delta t_0) \approx a(t_0)\big(1+H_0\Delta t_0\big), \qquad a(t_s+\Delta t_s) \approx a(t_s)\big(1+H(z_1)\Delta t_s\big),$$

so

$$\frac{a(t_0+\Delta t_0)}{a(t_s+\Delta t_s)} \approx \frac{a(t_0)}{a(t_s)}\Big[1+H_0\,\Delta t_0 - H(z_1)\,\Delta t_s\Big] = (1+z_1)\Big[1+H_0\,\Delta t_0 - \tfrac{H(z_1)}{1+z_1}\,\Delta t_0\Big].$$

Subtracting $1+z_1$ yields

$$\Delta z = H_0\,\Delta t_0\,(1+z_1) \;-\; H(z_1)\,\Delta t_0 = H_0\,\Delta t_0\big[(1+z_1) - E(z_1)\big], \tag{5}$$

where $H(z_1) = H_0\,E(z_1)$.

(d) In the CPL model with $w_0 = -1$ and $|w_a| \ll 1$,

$$E(z) \approx E_\Lambda(z) + \frac{3\,\Omega_{\mathrm{DE},0}}{2\,E_\Lambda(z)}\left[\ln(1+z) - \tfrac{z}{1+z}\right] w_a, \quad E_\Lambda(z) = \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\mathrm{DE},0}}.$$

The w_a–induced change in the drift is

$$\Delta z(w_a) - \Delta z(0) \approx -\,H_0\,\Delta t_0\,\frac{3\,\Omega_{\mathrm{DE},0}}{2\,E_\Lambda(z_1)}\left[\ln(1+z_1) - \tfrac{z_1}{1+z_1}\right] w_a.$$

At $z_1 = 1$, $\Omega_{m,0} = 0.3$, $\Omega_{\mathrm{DE},0} = 0.7$, $E_\Lambda(1) \approx 1.76$, $\ln 2 - \frac{1}{2} \approx 0.193$, $H_0 \approx 10^{-10}\,\mathrm{yr}^{-1}$, $\Delta t_0 = 10\,\mathrm{yr}$,

$$H_0 \Delta t_0 \approx 10^{-9}, \quad \frac{3\,\Omega_{\mathrm{DE},0}}{2E_\Lambda(1)} \times 0.193 \approx 0.115,$$

so

$$|\Delta z(w_a) - \Delta z(0)| \approx 1.15 \times 10^{-10}\,|w_a|,$$

and requiring $\gtrsim 10^{-11}$ precision gives

$$|w_a| \gtrsim \frac{10^{-11}}{1.15 \times 10^{-10}} \sim 0.1.$$

Problem 2

(a) Use the first law of thermodynamics $T\,dS = dU + P\,dV$ to show that the entropy in a expanding patch of volume V is conserved for a fluid with density and pressure ρ, P satistying the continuity equation

$$d\rho + (\rho + P)\frac{dV}{V} = 0 \tag{18}$$

(you may assume that the chemical potential vanishes).
Show that the entropy density is given by

$$s = \frac{\rho + P}{T}\,. \tag{19}$$

Hint: consider variations of thermodynamic quantities that can be set to zero independently.

Consider a hot big bang universe at temperatures $T \gg 100$ GeV consisting of the Standard Model (SM) and, in addition, a relativistic "dark" scalar particle ϕ. For $T > T_d$ the SM particles and ϕ are in kinetic equilibrium, but for $T < T_d$ the particle ϕ is decoupled from the SM. ϕ self-interacts through the processes $\phi\phi \leftrightarrow \phi\phi$ and $\phi\phi \leftrightarrow \phi\phi\phi$.

(b) What is the chemical potential of ϕ? Find the ratio of the entropy densities before decoupling,

$$\xi(T) \equiv \frac{s_{\rm SM}}{s_\phi} \tag{20}$$

in terms of the thermal degrees of freedom $g^{\rm SM}$ and g^ϕ. Estimate the magnitude of $\xi(T)$ and briefly describe its evolution after decoupling.

(c) Some time after decoupling from the Standard Model, ϕ will become non-relativistic, but it will be kept in a thermal state for some time due to its self-interactions. Defining $x \equiv m_\phi/T_\phi$, show that when $x \gg 1$ we get

$$\rho_\phi \simeq m_\phi n_\phi(T) \tag{21}$$

where n_ϕ is the number density of ϕ. Hence, derive following expression for the entropy density

$$s_\phi \simeq \frac{m_\phi^3}{(2\pi)^{3/2}} x^{-1/2}\, e^{-x}\,. \tag{22}$$

Show that long after decoupling from the Standard Model (but still assuming that ϕ is in a thermal state), the ratio of temperatures between the photons and the ϕ particles is

$$\frac{T_\gamma}{T_\phi} \simeq k \left(\frac{\xi}{g_{eff}^{\rm SM}}\right)^{1/3} x^{5/6}\, e^{-x/3}, \tag{23}$$

where k is a numerical factor to be determined, and $g_{eff}^{\rm SM}$ is the temperature-dependent effective number of degrees of freedom, defined as $g_{eff}^{\rm SM}(T_\gamma) \equiv \sum_i g_i \left(\frac{T_i}{T_\gamma}\right)^3$, where the index i runs over all the Standard Model particles.

Hint:

$$\int_0^\infty y^2 e^{-y^2}\,{\rm d}y = \frac{\sqrt{\pi}}{4}.$$

(d) As long as the ϕ particles interact with each other, does their temperature decrease faster or slower than the photon temperature as the universe expands?
After all the self-interactions between ϕ particles freeze out, argue why ϕ still obeys a Bose-Einstein distribution for non-relativistic particles, and find how its effective temperature T_ϕ^{eff} scales with the scale factor $a(t)$. How does T_ϕ^{eff} scale with the T_γ?

Solution

(a) Consider a comoving volume $V \propto a^3(t)$ containing a fluid with energy density ρ, pressure P, and temperature T, and assume the chemical potential vanishes, $\mu = 0$. The first law of thermodynamics reads

$$dU + P\,dV = T\,dS, \tag{24}$$

with

$$U = \rho V, \quad S = s\,V. \tag{25}$$

Energy–momentum conservation in an FRW universe gives the continuity equation

$$d\rho + (\rho + P)\,\frac{dV}{V} = 0. \tag{26}$$

From (24) and $U = \rho V$ we get

$$d(\rho V) + P\,dV = T\,dS \implies V\,d\rho + (\rho + P)\,dV = T\,dS. \tag{27}$$

Substituting (26) into this expression,

$$T\,dS = 0 \implies dS = 0 \implies S = \text{const.} \tag{28}$$

Thus the total entropy in a comoving volume is conserved.

Next, write $S = sV$, so

$$dS = V\,ds + s\,dV. \tag{29}$$

Then

$$V\,d\rho + (\rho + P)\,dV = T\,(V\,ds + s\,dV). \tag{30}$$

Rearrange to

$$V\,(d\rho - T\,ds) + (\rho + P - T\,s)\,dV = 0. \tag{31}$$

This must stay true when we do not vary the volume of the system, i.e. $dV = 0$, yielding

$$d\rho = T\,ds. \tag{32}$$

Now, if we plug this into (31) and consider a system whose size varies, we get

$$s = \frac{\rho + P}{T}. \tag{33}$$

(b) A relativistic scalar ϕ self–interacts via $\phi\phi \leftrightarrow \phi\phi$ and $\phi\phi \leftrightarrow \phi\phi\phi$, so there is no conserved ϕ–number. Chemical equilibrium therefore imposes

$$\mu_\phi = 0. \tag{34}$$

For $T > T_d$, the SM and ϕ sectors share the same temperature T. A relativistic species i has entropy density

$$s_i = \frac{2\pi^2}{45}\, g^i_{*s}\, T^3, \tag{35}$$

where g^i_{*s} counts entropy degrees of freedom. Thus

$$s_{\rm SM} = \frac{2\pi^2}{45}\, g^{\rm SM}_{*s}\, T^3, \quad s_\phi = \frac{2\pi^2}{45}\, g^{\phi}_{*s}\, T^3. \tag{36}$$

Just before decoupling ($T \gtrsim T_d$),

$$\xi \equiv \frac{s_{\rm SM}}{s_\phi} = \frac{g^{\rm SM}_{*s}}{g^{\phi}_{*s}}. \tag{37}$$

After decoupling ($T < T_d$), each sector conserves its own comoving entropy,

$$s_{\rm SM}\, a^3 = \text{const}, \quad s_\phi\, a^3 = \text{const}, \tag{38}$$

so the ratio ξ remains fixed at

$$\xi(T < T_d) = \frac{g^{\rm SM}_{*s}(T_d)}{g^{\phi}_{*s}}. \tag{39}$$

(c) We begin with the entropy density of the ϕ–sector,

$$s_\phi \;=\; \frac{\rho_\phi + P_\phi}{T_\phi} \;\simeq\; \frac{\rho_\phi}{T_\phi} \quad \big(T_\phi \ll m_\phi\big), \tag{40}$$

since for a non–relativistic gas $P_\phi \ll \rho_\phi$. The energy density is

$$\rho_\phi = \frac{g_\phi}{2\pi^2} \int_0^\infty dp\, p^2\, \frac{\sqrt{p^2 + m_\phi^2}}{\exp\!\big(\sqrt{p^2 + m_\phi^2}/T_\phi\big) - 1}\,. \tag{41}$$

Define the dimensionless variable

$$x \;\equiv\; \frac{m_\phi}{T_\phi} \;\gg 1. \tag{42}$$

In this limit we expand $\sqrt{p^2 + m_\phi^2} \approx m_\phi + p^2/(2m_\phi)$ and pull out the Boltzmann factor $e^{-m_\phi/T_\phi} = e^{-x}$:

$$\rho_\phi \simeq \frac{g_\phi}{2\pi^2}\, m_\phi \int_0^\infty dp\, p^2\, \exp\!\Big(-x - \frac{p^2}{2m_\phi T_\phi}\Big) \;=\; m_\phi\, n_\phi, \tag{43}$$

$$n_\phi \equiv \frac{g_\phi}{2\pi^2} \int_0^\infty dp\, p^2 \exp\!\Big(-x - \frac{p^2}{2m_\phi T_\phi}\Big) = g_\phi \Big(\tfrac{m_\phi\, T_\phi}{2\pi}\Big)^{3/2} e^{-x}. \tag{44}$$

Therefore

$$s_\phi \simeq \frac{\rho_\phi}{T_\phi} = \frac{m_\phi\, n_\phi}{T_\phi} = \frac{m_\phi^3}{(2\pi)^{3/2}}\, x^{-1/2}\, e^{-x}. \tag{45}$$

The SM (photon) sector has entropy density

$$s_{\rm SM} = \frac{2\pi^2}{45}\, g^{\rm SM}_{\rm eff}(T_\gamma)\, T_\gamma^3. \tag{46}$$

After decoupling at $T = T_d$, each sector conserves its comoving entropy:

$$s_\phi\, a^3 = \text{const}, \quad s_{\rm SM}\, a^3 = \text{const} \quad \Longrightarrow \quad \xi \equiv \frac{s_{\rm SM}}{s_\phi} = \frac{g_{\rm eff}^{\rm SM}(T_d)}{g_\phi} = \text{const}.$$

Equate (46) to $\xi\, s_\phi$ using (45):

$$\frac{2\pi^2}{45}\, g_{\rm eff}^{\rm SM}\, T_\gamma^3 = \xi\, \frac{m_\phi^3}{(2\pi)^{3/2}}\, x^{-1/2} e^{-x}, \tag{47}$$

$$T_\gamma^3 = \xi\, \frac{m_\phi^3}{(2\pi)^{3/2}}\, \frac{45}{2\pi^2\, g_{\rm eff}^{\rm SM}}\, x^{-1/2} e^{-x}. \tag{48}$$

Finally, since $T_\phi = m_\phi / x$,

$$\frac{T_\gamma}{T_\phi} = \left[\xi\, \frac{m_\phi^3}{(2\pi)^{3/2}} \frac{45}{2\pi^2\, g_{\rm eff}^{\rm SM}}\, x^{-1/2} e^{-x}\right]^{1/3} \frac{x}{m_\phi} = k \left(\frac{\xi}{g_{\rm eff}^{\rm SM}}\right)^{1/3} x^{5/6} e^{-x/3}, \tag{49}$$

$$k \equiv \left[\tfrac{45}{2^{5/2}\pi^{7/2}}\right]^{1/3}. \tag{50}$$

(d) As long as the ϕ particles continue to scatter among themselves, they remain in true thermal equilibrium at a genuine thermodynamic temperature T_ϕ. After self–interactions freeze out, their distribution "freezes in" as a Bose–Einstein form characterized by an effective temperature $T_\phi^{\rm eff}$.

During the interacting phase, when T_ϕ is still well-defined, (49) holds, and given that T_γ is exponentially related to T_ϕ, T_ϕ changes very slowly as T_γ varies.
Once ϕ–ϕ interactions cease, the physical 3-momentum redshifts as $p \propto a^{-1}$ as the universe expands. For non-relativistic ϕ particles with energy $E = p^2/2m \propto a^{-2}$, we define T^{eff} such that the ratio if variables appearing in the Bose-Einstein distribution, $\frac{E}{T^{eff}}$, stays constant and therefore

$$T_\phi^{\rm eff} \propto a^{-2}, \qquad T_\phi^{\rm eff} \propto T_\gamma^2. \tag{51}$$

Problem 3

In this exercise we would like to study the evolution during *early* matter domination of modes of density fluctuations of the cold dark matter component which entered the horizon before matter-radiation equality η_{eq}. Recall the linearized Einstein equations

$$-\Delta\Phi + 3\frac{a'}{a}\Phi' + 3\frac{a'^2}{a^2}\Phi = -4\pi G a^2 \cdot \sum_\lambda \delta\rho_\lambda, \tag{52}$$

$$\Phi' + \frac{a'}{a}\Phi = -4\pi G a^2 \cdot \sum_\lambda [(\rho + p)v]_\lambda, \tag{53}$$

$$\Phi'' + 3\frac{a'}{a}\Phi' + \left(2\frac{a''}{a} - \frac{a'^2}{a^2}\right)\Phi = 4\pi G a^2 \cdot \sum_\lambda \delta p_\lambda, \tag{54}$$

where λ labels the components of the cosmic fluid and the dash indicates the derivative with respect to the conformal time. The Newtonian potential in momentum space at radiation domination was computed in class to be ($u_s^{rad} = \sqrt{3}$)

$$\Phi_\gamma(\eta) = -3\Phi_{(i)} \cdot \frac{1}{(u_s^{rad}k\eta)^2}\left[\cos(u_s^{rad}k\eta) - \frac{\sin(u_s^{rad}k\eta)}{u_s^{rad}k\eta}\right], \tag{55}$$

while the sub-horizon adiabatic mode of cold dark matter fluctuations during radiation domination were computed in the homework to be:

$$\delta_{CDM}(\eta) = -9\Phi_{(i)}\left[\log\left(\frac{k\eta}{\sqrt{3}}\right) + \gamma_E - \frac{1}{2}\right], \tag{56}$$

with η the conformal time and γ_E the Euler-Mascheroni constant.

(a) Using the first Einstein equation (52), compute Φ_{CDM}, the correction to the Newtonian potential Φ_γ generated by dark matter perturbations, deep inside the horizon during radiation domination. Show that this contribution is dominant at time of radiation-matter equality.

Hint 1: Argue why all terms with conformal time derivatives can be neglected.

The linearized covariant conservation of the energy-momentum tensor can be written in the form

$$\delta\rho'_\lambda + 3\frac{a'}{a}(\delta\rho_\lambda + \delta p_\lambda) + (\rho_\lambda + p_\lambda)\left(\nabla^2 v_\lambda - 3\Phi'\right) = 0, \tag{57}$$

$$[(\rho_\lambda + p_\lambda)v_\lambda]' + 4\frac{a'}{a}(\rho_\lambda + p_\lambda)v_\lambda + \delta p_\lambda + (\rho_\lambda + p_\lambda)\Phi = 0, \tag{58}$$

Perturbation of the radiation medium induced by Φ_{CDM} during matter domination are small, and therefore can be neglected for the rest of the computations.

(b) Starting from equations (57) and (58), derive the evolution equations for the CDM density contrast δ_{CDM} in a matter dominated background at leading order in $\frac{1}{k\eta}$.

Hint 2: recall the relation for background densities $\rho'_\lambda = -3\frac{a'}{a}(\rho_\lambda + p_\lambda)$.

Remark: The treatment of dark matter perturbations during matter domination seen in class, which only required to solve (54), does not apply here, since for $\eta \sim \eta_{eq}$, the universe is still a multi-component fluid.

The equation derived at the previous point is a homogeneous second order ordinary differential equation in δ_{CDM}. In order to solve this equation, it is convenient to introduce the variable $x(\eta) := \frac{a(\eta)}{a_{eq}}$; with this variable, the evolution equation for δ_{CDM} becomes:

$$x(x+1)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta_{CDM} + \left(1+\frac{3}{2}x\right)\frac{\mathrm{d}}{\mathrm{d}x}\delta_{CDM} - \frac{3}{2}\delta_{CDM} = 0, \tag{59}$$

where we neglected baryon contribution to the energy density, while keeping the radiation component contribution for the evolution of the Hubble parameter.

(c) Give a physical reason for why we can expect one solution of (59) to be linear in x.

For a homogeneous second order differential equation, once you know a solution $y^{(1)}(x)$ you may find the second one with the ansatz $y^{(2)}(x) = q(x) \cdot y^{(1)}(x)$, and solving a *first* order differential equation for $q'(x)$.

Find then the most general solution to the differential equation (59).

Hint 3: You may find useful the solution to the following integral

$$\int_{-\infty}^{x} \frac{dt}{t\sqrt{1+t}\left(1+\frac{3}{2}t\right)^2} = \ln\left(\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right) + 6\frac{\sqrt{1+x}}{2+3x}. \tag{60}$$

(d) Determine the physical solution for $\delta_{CDM}(\eta)$ by imposing the correct initial conditions at $x \ll 1$. Comment on the late time expansion of this solution.

Solution

(a) The 0−th component of the linearized Einstein equations reads

$$-\Delta\Phi + 3\frac{a'}{a}\Phi' + 3\frac{a'^2}{a^2}\Phi = -4\pi G a^2 \cdot \sum_\lambda \delta\rho_\lambda, \tag{61}$$

where dash is again the derivative with respect to conformal time; once we take the Fourier transform on the spatial slices:

$$k^2\Phi + 3\frac{a'}{a}\Phi' + 3\frac{a'^2}{a^2}\Phi = -4\pi G a^2 \cdot \sum_\lambda \delta\rho_\lambda. \tag{62}$$

For the CDM component,

$$k^2\Phi_{CDM} + 3\frac{a'}{a}\Phi'_{CDM} + 3\frac{a'^2}{a^2}\Phi_{CDM} = -4\pi G a^2 \cdot \delta\rho_{CDM} = -4\pi G a^2 \cdot \rho_{CDM}\delta_{CDM}. \tag{63}$$

Deep inside the horizon $k\eta \gg 1$, so we can neglect all conformal time derivatives and

$$\boxed{\Phi_{CDM} = -\frac{a^2(\eta)}{k^2} 4\pi G \rho_{CDM}\delta_{CDM}(\eta).} \tag{64}$$

A similar equation holds for the Newtonian potential Φ_γ, reported in (55). Since $\rho_{CDM} \propto \frac{1}{a^3}$, while $\rho_\gamma \propto \frac{1}{a^4}$, and neglecting logarithmic terms coming from δ_{CDM}, within radiation domination we have

$$\begin{cases} \Phi_\gamma \propto \frac{1}{a^2} \\ \Phi_{CDM} \propto \frac{1}{a} \end{cases} \implies \Phi_{CDM} \ll \Phi \iff \eta < \eta_{eq}. \tag{65}$$

Comment: perturbation of the radiation medium can be neglected during matter dominations[1]

During matter domination, where Φ_{CDM} is the dominant contribution to the Newtonian potential, the zeroth component of linearized Einstein equations gives (always for modes well inside the horizon)

$$\delta_{tot} \sim \frac{k^2}{4\pi G\rho a^2}\Phi_{CDM} \sim \frac{k^2}{a^2H^2}\Phi_{CDM} \gg \Phi_{CDM}, \tag{66}$$

since modes inside the sound horizon are well inside the cosmological horizon. Fourier transform of equation (58) reads:

$$[(1+w_\lambda)v_\lambda]' + \frac{a'}{a}(1-3w_\lambda)(1+w_\lambda)v_\lambda + u^2_{s,\lambda}\delta_\lambda = -(1+w_\lambda)\Phi, \tag{67}$$

where we used the definitions of the sound speed $u^2_{s,\lambda} = \frac{\delta p_\lambda}{\delta\rho_\lambda}$ and of the parameter of the equation of state $w_\lambda = \frac{p_\lambda}{\rho_\lambda}$, together with the identity for *background* densities $\rho'_\lambda = -3\frac{a'}{a}\rho_\lambda(1+w_\lambda)$. For modes well inside the horizon (so to neglect time-derivative at first approximation) of the relativistic component $w_\gamma = \frac{1}{3}$, we obtain, during matter domination

$$\delta_\gamma \sim \Phi_{CDM} \ll \frac{k^2}{a^2H^2}\Phi_{CDM} \sim \delta_{tot}, \tag{68}$$

so we can safely neglect the effect of perturbations of the relativistic medium for further computations.

[1] These comments in the solution are meant to justify assumptions and results given in the problem text, but students were not required to comment during the exam and can be omitted at a first reading.

(b) In order to derive the evolution equation for δ_{CMB} during matter domination, we need to Fourier transform equations (57) and (58), and write it in momentum space:

$$\delta\rho'_\lambda + 3\frac{a'}{a}\left(\delta\rho_\lambda + \delta p_\lambda\right) - \left(\rho_\lambda + p_\lambda\right)\left(k^2 v_\lambda + 3\Phi'\right) = 0. \tag{69}$$

$$[(1+w_\lambda)v_\lambda]' + \frac{a'}{a}(1-3w_\lambda)(1+w_\lambda)v_\lambda + u_{s,\lambda}^2\delta_\lambda = -(1+w_\lambda)\Phi, \tag{70}$$

Now we want to manipulate equations (69) and (70) so to have a differential equation for δ_λ in terms of known quantities. Let's start by noticing

$$\delta'_\lambda = \left(\frac{\delta\rho_\lambda}{\rho_\lambda}\right)' = \frac{\delta\rho'_\lambda}{\rho_\lambda} - \frac{\rho'_\lambda}{\rho_\lambda}\delta_\lambda \implies \delta\rho'_\lambda = \rho_\lambda\delta'_\lambda + \rho'_\lambda\delta_\lambda.$$

Using the hint we have

$$\rho'_\lambda = -3\frac{a'}{a}\rho_\lambda\left(1+w_\lambda\right) \implies \rho'_\lambda\delta_\lambda = -3\frac{a'}{a}\delta\rho_\lambda\left(1+w_\lambda\right) = -3\frac{a'}{a}\left(\delta\rho_\lambda + \delta p_\lambda\right),$$

so equation (69) becomes

$$\delta'_\lambda - \left(1+w_\lambda\right)\left(k^2 v_\lambda + 3\Phi'\right) = 0, \tag{71}$$

which, for the specific case of Cold Dark Matter ($w_{CDM} = 0$), becomes

$$\delta'_{CDM} - k^2 v_{CDM} = 3\Phi', \tag{72}$$

and its time derivative is

$$\delta''_{CDM} - k^2 v'_{CDM} = 3\Phi'' \ll k^2\Phi_{CDM}, \tag{73}$$

Equation (70) instead, for CDM perturbations, becomes ($u_{s,CDM}^2 = w_{CDM} = 0$)

$$v'_{CDM} + \frac{a'}{a}v_{CDM} = -\Phi_{CDM}. \tag{74}$$

After $\eta \gtrsim \eta_{eq}$ we observe that $\Phi_{CDM}(\eta)$ evolves very slowly, since $\rho_{CDM} \sim \frac{1}{a^3}$ and δ_{CDM} is expected to grow as $\sim a$ from the study of matter perturbations in a matter dominated background. Therefore, we neglect time derivatives of Φ and substituting (72) and (73) into (74) we get

$$\boxed{\delta''_{CDM} + \frac{a'}{a}\delta'_{CDM} = -k^2\Phi_{CDM} = 4\pi G_N\rho_{CDM}a^2(\eta)\delta_{CDM}} \tag{75}$$

Comment: Rewriting the evolution equation for δ_{CDM} in terms of the variable $x(\eta) := \frac{a(\eta)}{a_{eq}}$

First of all, neglecting baryon contribution to non-relativistic energy density, Einstein's equations at radiation-matter equality give

$$\frac{8\pi}{3}G_N\rho_{CDM}\left(\eta_{eq}\right) = \frac{8\pi}{3}G_N\rho_\gamma\left(\eta_{eq}\right) = \frac{1}{2}H^2\left(\eta_{eq}\right) := \frac{1}{2}H_{eq}^2 \tag{76}$$

Moreover, $\rho_{CDM} \propto \frac{1}{a^3}$ while $\rho_\gamma \propto \frac{1}{a^4}$, so

$$\begin{aligned} 4\pi G_N\rho_{CDM}(\eta) = 4\pi G_N\rho_{CDM}(\eta_{eq})\cdot\frac{a_{eq}^3}{a^3} = \frac{3}{4}H_{eq}\cdot\frac{1}{x^3} \\ 4\pi G_N\rho_\gamma(\eta) = 4\pi G_N\rho_\gamma(\eta_{eq})\cdot\frac{a_{eq}^4}{a^4} = \frac{3}{4}H_{eq}\cdot\frac{1}{x^4} \end{aligned} \tag{77}$$

Therefore, Einstein's equation in terms of the new variable x read (remember that with conformal time, $H = \frac{a'}{a^2}$)

$$\frac{a'^2}{a^2} = \frac{1}{2} H_{eq}^2 a_{eq}^2 \left(\frac{1}{x} + \frac{1}{x^2}\right) \implies H(x) = \frac{H_{eq}}{\sqrt{2}} \cdot \frac{\sqrt{x+1}}{x^2}. \tag{78}$$

We can also transform the derivatives of conformal time to derivatives of x with the help of chain-rule:

$$\mathrm{d}x = \frac{\mathrm{d}a}{a_{eq}} = \frac{a'}{a_{eq}}\mathrm{d}\eta = \frac{a'}{a^2} x^2 a_{eq} \mathrm{d}\eta \implies \boxed{\frac{\mathrm{d}}{\mathrm{d}\eta} = H x^2 a_{eq} \frac{\mathrm{d}}{\mathrm{d}x}} \tag{79}$$

so the first derivative term of equation (75) becomes

$$\frac{a'}{a}\delta'_{CDM} = \frac{a'}{a} H x^2 a_{eq} \frac{\mathrm{d}}{\mathrm{d}x}\delta_{CDM} = H^2 x^3 a_{eq}^2 \frac{\mathrm{d}}{\mathrm{d}x}\delta_{CDM}. \tag{80}$$

For the second derivative:

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} &= \frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left(H x^2 a_{eq} \frac{\mathrm{d}}{\mathrm{d}x}\right) \\
&= a_{eq}^2 H x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(H x^2 \frac{\mathrm{d}}{\mathrm{d}x}\right) = \\
&= a_{eq}^2 H x^2 \left[H x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2Hx \frac{\mathrm{d}}{\mathrm{d}x} + x^2 \frac{\mathrm{d}H}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\right] = \\
&= a_{eq}^2 H^2 x^4 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + a_{eq}^2 H^2 x^3 \left[2 - \frac{1}{2} \cdot \frac{3x+4}{x+1}\right] \frac{\mathrm{d}}{\mathrm{d}x}
\end{aligned} \tag{81}$$

where we used

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}x} &= \frac{H_{eq}}{\sqrt{2}} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{x+1}}{x^2}\right) \\
&= -\frac{H_{eq}}{\sqrt{2}} \cdot \frac{3x+4}{2x^3\sqrt{x+1}} \\
&= -\frac{1}{2} H \cdot \frac{3x+4}{x(1+x)}.
\end{aligned} \tag{82}$$

Putting it all together, the differential equation in (75) becomes

$$a_{eq}^2 H^2 x^4 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta_{CDM} + a_{eq}^2 H^2 x^3 \left[2 - \frac{3x+4}{2(x+1)} + 1\right] \frac{\mathrm{d}}{\mathrm{d}x}\delta_{CDM} - \frac{3}{4} H_{eq}^2 a_{eq}^2 \cdot \frac{1}{x}\delta_{CDM} = 0$$

$$\boxed{x(x+1)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta_{CDM} + \left(1 + \frac{3}{2}x\right)\frac{\mathrm{d}}{\mathrm{d}x}\delta_{CDM} - \frac{3}{2}\delta_{CDM} = 0} \tag{83}$$

as was given in the main text.

(c) From the study of Jeans's instability that we have seen in class, you know that matter perturbations during matter dominations will grow linearly at some point. Therefore, a solution to the differential equation (59) should be (at least asymptotically) of the form

$$y^{(1)}(x) = A_1 x + B_1; \tag{84}$$

plugging it into (59) we easily get

$$A_1 = \frac{3}{2} B_1 \tag{85}$$

so the first homogeneous solution is:

$$y^{(1)}(x) = C_1 \left(1 + \frac{3}{2}x\right). \tag{86}$$

Making use of the suggestion given in point (c) (the q in the hint is more generally known as the Wronskian), we can write the second homogeneous solution as

$$y^{(2)}(x) = q(x) \cdot y^{(1)}(x), \tag{87}$$

and plugging it in equation (59) we find a first order differential equation for $q'(x)$:

$$\begin{aligned} A(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[q(x)\cdot y^{(1)}(x)\right] + B(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[q(x)\cdot y^{(1)}(x)\right] + C(x)\cdot q(x)\cdot y^{(1)}(x) = 0 \\ q''(x)A(x)y^{(1)}(x) + q'(x)\left[2A(x)y'^{(1)}(x) + B(x)y^{(1)}(x)\right] = 0 \\ \frac{\mathrm{d}q'}{q'} = \left[2\frac{y'^{(1)}(x)}{y^{(1)}(x)} + \frac{B(x)}{A(x)}\right]\mathrm{d}x. \end{aligned} \tag{88}$$

Substituting the values $A(x) = x(1+x)$ and $B(x) = 1 + \frac{3}{2}x$ relevant for our case, we can solve for $q'(x)$

$$\begin{aligned} \ln q' &= -\int \mathrm{d}x\, \frac{3}{\frac{3}{2}x+1} + \underbrace{\frac{1}{x(x+1)}}_{\frac{1}{x}-\frac{1}{x+1}} + \frac{3}{2}\frac{1}{(x+1)} = \\ &= -\left[2\ln\left(x+\frac{2}{3}\right) + \ln x + \frac{1}{2}\ln(x+1)\right] \end{aligned} \tag{89}$$

so, using Hint 3:

$$\begin{aligned} q(x) &= C_2 \int_{-\infty}^{x} \mathrm{d}t \frac{1}{t\sqrt{1+t}\left(\frac{3}{2}t+1\right)^2} = \\ &= C_2\left[\ln\left(\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right) + 6\frac{\sqrt{1+x}}{2+3x}\right]. \end{aligned} \tag{90}$$

The full solution to equation (59) is then

$$\boxed{\delta_{CDM}(x) = \left(1+\frac{3}{2}x\right)\left[C_1 + C_2\left(\ln\left(\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right) + 6\frac{\sqrt{1+x}}{2+3x}\right)\right]} \tag{91}$$

(d) In order to fix the constants C_1 and C_2, we need to match it with the solution for $\delta_{CDM}(\eta)$ in the radiation-dominated regime; expanding (91) for $x \ll 1$, we find:

$$\delta_{CDM}(x) = C_1 + C_2\left[\ln x - 2\ln 2 + 3\right] = C_1 + C_2\left[\ln\frac{\eta}{\eta_{eq}} - 2\ln 2 + 3\right], \tag{92}$$

where we also used that $a(\eta) \propto \eta$ during radiation domination. The behavior of $\delta_{CDM}(\eta)$ is given in the text (56):

$$\delta_{CDM}(\eta) = -9\Phi_{(i)}\left[\log\left(\frac{k\eta}{\sqrt{3}}\right) + \gamma_E - \frac{1}{2}\right], \tag{93}$$

so

$$C_2 = -9\Phi_{(i)} \tag{94}$$

and, at this level of approximation

$$C_1 = -9\Phi_{(i)} \left[\ln(k\eta_{eq}) + \gamma_E - \frac{7}{2} + \ln \frac{4}{\sqrt{3}} \right]. \tag{95}$$

At late times instead, the second homogeneous solution is suppressed, and

$$\boxed{\begin{aligned} \delta_{CDM}(\eta \gg \eta_{eq}) = C_1 \cdot \frac{3}{2} x &= -\frac{27}{2} \frac{a(\eta)}{a(\eta_{eq})} \Phi_{(i)} \left[\ln(k\eta_{eq}) + \gamma_E - \frac{7}{2} + \ln \frac{4}{\sqrt{3}} \right] \\ &\simeq -\frac{27}{2} \frac{a(\eta)}{a(\eta_{eq})} \Phi_{(i)} \ln(0.15 k\eta_{eq}) \end{aligned}} \tag{96}$$

We can see that the mechanism of generation of dark matter perturbation is quite efficient: compared to modes which enter the horizon at matter domination, those which enter at radiation domination get not only a logarithmic enhancement, but also an extra factor $\frac{3}{2}$, which would have not been obtained by simple matching of (56) at $\eta = \eta_{eq}$.