RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 9 Fall 2023

1. Escaping the Photon Sphere

(a) Because of the fact that the Schwarzschild metric is symmetric under $\theta \to \pi - \theta$, we deduce that orbits with $\theta = \frac{\pi}{2}$, will always keep θ constant. That makes intuitive sense: similarly to the geodesics on the sphere, the only geodesics on Schwarlzschild with constant θ are the ones that have $\theta = \frac{\pi}{2}$. That means that specifying ourselves to $\theta = \frac{\pi}{2}$ simplifies the calculations, and by rotational symmetry there is no loss of generality. Null geodesics with $\theta = \frac{\pi}{2}$ satisfy

$$0 = ds^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - \frac{2M}{r}\right)} + r^2 \mathrm{d}\phi^2 \,. \tag{1}$$

The Killing vectors of interest to us are the vector field that generates time translations, $K_t = \partial_t$, and the one that generates rotations in ϕ, $K_\phi = \partial_\phi$. For any Killing vector K^μ there is a conserved quantity given by $g_{\mu\nu} K^\mu \frac{dx^\nu}{d\lambda}$. We thus have the conserved energy

$$E = g_{\mu\nu} K_t^\mu \frac{dx^\nu}{d\lambda} = -\left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} \,, \tag{2}$$

and the conserved angular momentum

$$L = g_{\mu\nu} K_\phi^\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \,. \tag{3}$$

Now, as we did in Problem Set 6, let us act with ds^2 on two copies of the vector $V = \frac{dx^\mu}{d\lambda} \partial_\mu$. We obtain

$$\begin{aligned} -\left(1 - \frac{2M}{r}\right) \left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\phi}{d\lambda}\right)^2 = 0 \\ -\frac{E^2}{\left(1 - \frac{2M}{r}\right)} + \frac{1}{\left(1 - \frac{2M}{r}\right)} \left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} = 0 \,. \end{aligned} \tag{4}$$

This equation can also be derived by the action formulation, noting that the Lagrangian $\mathcal{L} = g_{\mu\nu} \dot{x}^\nu \dot{x}^\mu$ is invariant under λ translations. This gives us the conservation of the 'Hamiltonian' $\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \dot{x}^\mu - \mathcal{L} = g_{\mu\nu} \dot{x}^\nu \dot{x}^\mu$. Since it's a photon, this is equal to 0, which gives equation (4).

Solving for E we find

$$\frac{E^2}{2} = \frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \equiv \frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 + V(r) \,. \tag{5}$$

By plotting the effective potential $V(r)$ we see that it has a maximum at $r = 3M$, defining a barrier for the photon to surpass in order to be free from the pull of the black hole. The height of the maximum is

$$V(3M) = \frac{L^2}{54M^2}\,. \tag{6}$$

The condition on the energy of the photon is thus

$$\frac{E^2}{2} \geq \frac{L^2}{54M^2}\,. \tag{7}$$

(b) First we need to define what an "angle" is. In this context we mean the spatial angle, that therefore needs to be computed with the spatial part of the metric (which is possible in Schwartzschild geometry since there are no $dtdx^i$ components in the metric : we say that this metric foliates spacetime into spacelike slices). Recall the usual definition for the angle α between two vectors V and W

$$V \cdot W = |V||W| \cos\alpha. \tag{8}$$

In geometry the scalar product and norms must be computed with the metric, hence

$$V \cdot W = g_{ij}V^iV^j, \qquad |V| = \sqrt{g_{ij}V^iV^j}. \tag{9}$$

In our problem the angular direction generated by ∂_ϕ is perpendicular to the radial direction. We can thus compute the angle that the direction of propagation of a photon makes with the radial direction from the definition of $\sin\alpha$ in terms of a ratio of one of the sides of the right triangle with respect to its hypothenuse (we can also of course compute cosα by taking the dot product of V and ∂_r and check that $\cos^2\alpha + \sin^2\alpha = 1$).

$$\begin{aligned}\sin^2\alpha &= \frac{g_{\phi\phi}V^\phi V^\phi}{g_{\phi\phi}V^\phi V^\phi + g_{rr}V^rV^r}\,. \\ &= \frac{r^2\left(\frac{d\phi}{d\lambda}\right)^2}{r^2\left(\frac{d\phi}{d\lambda}\right)^2 + \frac{1}{1-\frac{2M}{r}}\left(\frac{dr}{d\lambda}\right)^2} \\ &= \frac{L^2}{E^2r^2}\left(1 - \frac{2M}{r}\right)\,,\end{aligned} \tag{10}$$

where we used that $\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{L^2}{r^2} + \frac{2ML^2}{r^3}$. The condition we found on the energy of the photon implies that

$$\sin^2\alpha \leq \frac{27M^2}{r^2}\left(1 - \frac{2M}{r}\right)\,, \tag{11}$$

as we wanted to prove. The fact that for $r < 2M$ the right hand side is a negative number also shows that photons below the event horizon cannot escape.

2. Orbiting Gargantua

(a) The argument goes exactly like in Problem Set 8. Because the observers are static,

$$\Delta\tau_A = \left(1 - \frac{2M}{r_A}\right)\Delta t\,, \qquad \Delta\tau_C = \left(1 - \frac{2M}{r_C}\right)\Delta t\,, \tag{12}$$

so that equating the Δt in both cases gives the relation between the time intervals that we were looking for.

(b) As argued in the previous problem, we have that

$$E = -\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}\,, \tag{13}$$

where for timelike geodesics we choose to normalize energy with respect to proper time. We are considering radial geodesics, such that ϕ and θ are fixed, but both t and r change

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}}\,. \tag{14}$$

By acting with this tensor on two copies of the vector ∂_τ, we get

$$\begin{aligned} 1 &= \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{1 - \frac{2M}{r}}\left(\frac{dr}{d\tau}\right)^2 \\ 1 &= \frac{1}{1 - \frac{2M}{r}}\left(E^2 - \left(\frac{dr}{d\tau}\right)^2\right)\,. \end{aligned} \tag{15}$$

Reshuffling terms, we get

$$d\tau = -\frac{\mathrm{d}r}{\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)}}\,, \tag{16}$$

where we took the negative root because Cooper is moving towards decreasing values of r. The absolute value of the finite proper time interval $\Delta\tau_C$ is then

$$\Delta\tau_C = -\int_{r_C}^{r_C - \Delta r_C} \frac{\mathrm{d}r}{\sqrt{E^2 - 1 + \frac{2M}{r}}}\,. \tag{17}$$

We want to check that this is finite as $r_C \to 2M$. It suffices to expand the integrand around that value. Specifically, the small dimensionless quantity is $1 - \frac{2M}{r}$. We obtain

$$\begin{aligned} \Delta\tau_C &\approx -\int_{r_C}^{r_C - \Delta r_C} \mathrm{d}r\left[\frac{1}{E} + \frac{1}{2E^3}\left(1 - \frac{2M}{r}\right)\right] \\ &= \frac{\Delta r_C}{E}\left(1 + \frac{1}{2E^2}\right) - \frac{M}{E^3}\log\left(\frac{r_C}{r_C - \Delta r_C}\right) \end{aligned} \tag{18}$$

where we are indicating by Δr_C the absolute value of the difference between the position of Cooper at the two different proper times.

This quantity is perfectly well behaved at the event horizon. Cooper does not observe anything special when he crosses $r_C = 2M$.

(c) Using the definition of energy, the coordinate time interval between the emission of two light signals by Cooper can be expressed as

$$\mathrm{d}t^2 = \frac{E^2}{\left(1 - \frac{2M}{r}\right)^2} d\tau^2 \,. \tag{19}$$

Using (16) we get

$$\mathrm{d}t = -\frac{E}{\left(1 - \frac{2M}{r}\right)\sqrt{E^2 - 1 + \frac{2M}{r}}} \mathrm{d}r \,. \tag{20}$$

Expanding near the horizon, we get

$$\begin{aligned} \Delta t_C &\approx -\int_{r_C}^{r_C - \Delta r_C} \left[\frac{1}{2E^2} + \frac{1}{1 - \frac{2M}{r}}\right] \mathrm{d}r \\ &= \left(1 + \frac{1}{2E^2}\right) \Delta r_C - 2M \log\left(\frac{r_C - \Delta r_C - 2M}{r_C - 2M}\right) , \end{aligned} \tag{21}$$

which is clearly divergent as $r_C \to 2M$. The coordinate time interval associated with Amelia's reception of the photons is simply given by

$$\Delta t_A = \Delta t_C + \Delta r_C \,. \tag{22}$$

Amelia is at rest in this coordinate frame, so

$$\begin{aligned} \Delta \tau_A &= \sqrt{1 - \frac{2M}{r_A}} \Delta t_A \\ &= \sqrt{1 - \frac{2M}{r_A}} \left[\frac{\Delta r_C}{2E^2} - 2M \log\left(\frac{r_C - \Delta r_C - 2M}{r_C - 2M}\right)\right] , \end{aligned} \tag{23}$$

which diverges as Cooper crosses the horizon.

3. Light Deflection in Scalar Gravity

(a) Let us start from the time component of the equations of motion

$$\frac{d}{d\tau}\gamma(v) = -\frac{dt}{d\tau}\frac{dx^i}{d\tau}\partial_i \Phi(x) \,, \tag{24}$$

where we've used that $\partial_t \Phi(x) = 0$. Using that $\frac{dt}{d\tau} = \gamma(v)$ we can write (24) as

$$\begin{aligned} \gamma(v)\frac{d}{dt}\gamma(v) &= -\,\gamma(v)^2 \vec{v} \cdot \vec{\nabla}\Phi(x) \\ \vec{v} \cdot \vec{\nabla}\Phi(x) &= -\,\frac{v\dot{v}}{1 - v^2} \,. \end{aligned} \tag{25}$$

The spatial components of the equations of motion read, in terms of physical velocity,

$$\begin{aligned} \gamma(v)\frac{d}{dt}(v^i \gamma(v)) &= -(\partial_i \Phi(x) + \gamma(v)^2 v^i v^j \partial_j \Phi(x)) \\ \frac{\dot{v}^i}{1 - v^2} + \frac{v^i v \dot{v}}{(1 - v^2)^2} &= -\partial_i \Phi(x) - \frac{v^i \vec{v} \cdot \vec{\nabla}\Phi(x)}{1 - v^2} \,. \end{aligned} \tag{26}$$

Using (25) we get

$$\dot{v}^i = -(1-v^2)\partial_i \Phi(x)\,. \tag{27}$$

From this differential equation it is clear that if a particle starts by moving at the speed of light, $v = 1$, it stays at that speed at all times, independently of the presence of the gravitational field, and the direction of its motion is not affected.

(b) Varying the action with respect to the Lagrange multiplier $E(\lambda)$, we get

$$\delta_E S = \frac{1}{2}\int \mathrm{d}\lambda \left[-\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{E(\lambda)^2} - m^2 e^{2\Phi(x(\lambda))}\right]\delta E\,. \tag{28}$$

Imposing the vanishing of this variation for any δE we get

$$E(\lambda) = \frac{1}{m}e^{-\Phi(x(\lambda))}\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,. \tag{29}$$

The action with $E(\lambda)$ put on-shell is then

$$S' = -m\int \mathrm{d}\lambda\; e^{\Phi(x(\lambda))}\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,. \tag{30}$$

We then vary the action with respect to x.

$$\delta_x S' = -m\int \mathrm{d}\lambda\; e^{\Phi(x(\lambda))}\left[\partial_\mu\Phi(x(\lambda))\delta x^\mu\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma} + \frac{-\eta_{\mu\nu}\dot{x}^\mu\delta\dot{x}^\nu}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}}\right] \tag{31}$$

Let us focus on the second term in the brackets.Integrating it by parts, the derivative with respect to λ is going to hit the exponential, the square root and $\dot{x}^\mu$. We get

$$\begin{aligned} &m\int \mathrm{d}\lambda\; e^{\Phi(x(\lambda))}\frac{\dot{x}_\mu\delta\dot{x}^\mu}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} \\ &\quad= -m\int \mathrm{d}\lambda\; e^{\Phi(x(\lambda))}\left[\frac{\ddot{x}_\mu}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} + \frac{\dot{x}_\mu\eta_{\alpha\beta}\ddot{x}^\alpha\dot{x}^\beta}{(-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma)^{\frac{3}{2}}} + \frac{\partial_\nu\Phi(x(\lambda))\dot{x}^\nu\dot{x}_\mu}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}}\right]\delta x^\mu \end{aligned} \tag{32}$$

The second term in the brackets is null because a particle's four-acceleration is always orthogonal to its four-velocity (prove it). The equations of motion read

$$\begin{aligned} \partial_\mu\Phi(x(\lambda))\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma} + \frac{\ddot{x}_\mu}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} + \frac{\partial_\nu\Phi(x(\lambda))\dot{x}^\nu\dot{x}_\mu}{\sqrt{-\eta_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma}} = 0 \\ (-U^\alpha U_\alpha)\partial_\mu\Phi(x) + \frac{dU_\mu}{d\tau} + \partial_\nu\Phi(x)U^\nu U_\mu = 0\,, \end{aligned} \tag{33}$$

where we've used that λ is affinely related to τ. By using the fact that we are considering timelike geodesics ($U^\alpha U_\alpha = -1$) and reshuffling, we can write it as required

$$\frac{dU^\mu}{d\tau} = -\left(\eta^{\mu\nu} + U^\nu U^\mu\right)\partial_\nu\Phi(x) \tag{34}$$

Alternatively, you can also compute the equation of motions for x^μ before replacing $E(\lambda)$, and replacing its expression directly there. It should give the same result.