RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 13 — Fall 2022

1. Diffeomorphism invariance and the Conservation of $T^{\mu\nu}$

(a) The variation of the metric is defined as

$$\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x)\,, \tag{1}$$

where we will keep writing arguments explicitly where it will be pedagogically helpful. At the same time we have that, under a coordinate transformation

$$g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta}(x')\,. \tag{2}$$

Now let us focus on an infinitesimal diffeomorphism $x'^{\mu} = x^{\mu} - \xi^{\mu}$. The transformation becomes

$$g_{\mu\nu}(x) = \left(\delta^{\alpha}_{\mu} - \partial_{\mu}\xi^{\alpha}\right)\left(\delta^{\beta}_{\nu} - \partial_{\nu}\xi^{\beta}\right)\left(g'_{\alpha\beta}(x) - \xi^{\gamma}\partial_{\gamma}g'_{\alpha\beta}(x)\right)\,, \tag{3}$$

where we Taylor expanded the metric on the right hand side around x. Keeping only terms that are linear in ξ we get

$$g_{\mu\nu}(x) = g'_{\mu\nu}(x) - \partial_{\mu}\xi^{\alpha}g_{\alpha\nu}(x) - \partial_{\nu}\xi^{\beta}g_{\mu\beta}(x) - \xi^{\gamma}\partial_{\gamma}g_{\mu\nu}\,, \tag{4}$$

where we have used that $g'_{\mu\nu}(x) = g_{\mu\nu}(x) + O(\xi)$. We thus have

$$\begin{aligned}
\delta g_{\mu\nu} &= \left(\partial_{\mu}(g^{\alpha\rho}\xi_{\rho})g_{\alpha\nu} + \partial_{\nu}(g^{\beta\rho}\xi_{\rho})g_{\mu\beta} + \xi^{\rho}\partial_{\rho}g_{\mu\nu}\right)\\
&= \left(\partial_{\mu}g^{\alpha\rho}\xi_{\rho}g_{\alpha\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}g^{\beta\rho}\xi_{\rho}g_{\mu\beta} + \partial_{\nu}\xi_{\mu} + \xi^{\rho}\partial_{\rho}g_{\mu\nu}\right)\\
&= \left(2\partial_{(\mu}\xi_{\nu)} + \xi_{\rho}(g_{\alpha\nu}\partial_{\mu}g^{\alpha\rho} + g_{\mu\beta}\partial_{\nu}g^{\beta\rho} + \partial^{\rho}g_{\mu\nu})\right)\\
&= \left(2\partial_{(\mu}\xi_{\nu)} - \xi_{\rho}(g^{\rho\tau}\partial_{\mu}g_{\nu\tau} + g^{\rho\tau}\partial_{\nu}g_{\mu\tau} - \partial^{\rho}g_{\mu\nu})\right)\\
&= 2\left(\partial_{(\mu}\xi_{\nu)} - \xi_{\rho}\Gamma^{\rho}_{\mu\nu}\right)\\
&= 2\nabla_{(\mu}\xi_{\nu)}\,,
\end{aligned} \tag{5}$$

where, to go from the third to the fourth step, we've used $\partial_{\gamma}g^{\alpha\beta} = -g^{\alpha\alpha'}g^{\beta\beta'}\partial_{\gamma}g_{\alpha'\beta'}$.

(b) Consider a generic action

$$S = \int d^n x\sqrt{-g}\mathcal{L}(g_{\mu\nu}, \Phi)\,, \tag{6}$$

where Φ is a generic set of matter fields. Consider how the action changes under an infinitesimal diffeomorphism.

$$\begin{aligned}
\delta_{\mathrm{d}}S &= \int d^n x\;\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}\delta_{\mathrm{d}}g_{\mu\nu}\\
&= \frac{1}{2}\int d^n x\;\sqrt{-g}\;T^{\mu\nu}\;\delta_{\mathrm{d}}g_{\mu\nu}\\
&= \int d^n x\;\sqrt{-g}\;T^{\mu\nu}\;\nabla_{(\mu}\xi_{\nu)}\,,
\end{aligned} \tag{7}$$

where we've recognized the stress tensor $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}$. Now, integrating by parts,

$$\delta_{\rm d} S = \int d^n x\ \xi_\nu \sqrt{-g}\nabla_\mu T^{\mu\nu}\,. \tag{8}$$

The theory is diffeomorphism invariant if this vanishes for all infinitesimal ξ_ν . That happens only if $\nabla_\mu T^{\mu\nu} = 0$, as we wanted to prove.

2. Frame Dragging

(a) The sphere of mass M corresponds to a density distribution of $\rho(r) = \frac{M}{4\pi R^2}\delta(r-R)$. The energy momentum tensor is given by $T_{\mu\nu} = \rho U_\mu U_\nu$. The cartesian entries of the position vector describing the spherical shell reads

$$x^\mu = (t, R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)\ . \tag{9}$$

The four-velocity is thus

$$U^\mu = (1, -R\Omega\sin\theta\sin\phi, R\Omega\sin\theta\cos\phi, 0)\ , \tag{10}$$

where we used that the only time dependance is in $\phi(t) = \Omega t$ and we are neglecting higher orders in Ω since we are considering a slowly moving mass shell. The cartesian entries of the stress energy tensor that are linear in the velocity are thus

$$\begin{aligned} T_{00} &= \frac{M}{4\pi R^2}\delta(r-R)\,,\\ T_{01} &= -\Omega R\sin\theta\sin\phi\frac{M}{4\pi R^2}\delta(r-R)\,,\\ T_{02} &= \Omega R\sin\theta\cos\phi\frac{M}{4\pi R^2}\delta(r-R)\,, \end{aligned} \tag{11}$$

(b) Clearly, the T_{ij} terms are quadratic in v.

(c) Given that T_{tt}, T_{ti} are the only nonvanishing components, we can repackage the metric perturbation into a 4-vector $A_\mu = \bar{h}_{t\mu}$. Its equations of motion in the Lorenz gauge are

$$\Box A_\mu = \Box\bar{h}_{t\mu} = -16\pi T_{t\mu} = -J_\mu\,, \tag{12}$$

mirroring Maxwell's equations, with $J_\mu = 16\pi T_{t\mu}$. We can then define a gravito-electric and gravito-magnetic field

$$\begin{aligned} \vec{G}_i &\equiv \frac{1}{4}\left(\partial_i A_0 - \partial_0 A_i\right)\\ \vec{H}_i &\equiv \left(\varepsilon_{ijk}\partial_j A_k\right)\ . \end{aligned} \tag{13}$$

The conventional factor of $\frac{1}{4}$ will become more justified later.[1]

[1] Multiple conventions are present in the literature. In this case, as in Carroll's, fields are rescaled to provide familiar expression for Lorentz force. However this changes the form of the field equations.

(d) Let's start with the geodesic equation for the physical velocity

$$\frac{dv^i}{dt} = -\Gamma^i_{\mu\nu} v^\mu v^\nu + v^i \Gamma^t_{\mu\nu} v^\mu v^\nu \tag{14}$$

With Christoffel symbols being

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\alpha'}\left(\partial_\beta h_{\alpha'\gamma} + \partial_\gamma h_{\alpha'\gamma} - \partial_{\alpha'} h_{\beta\gamma}\right)$$

one can quickly identify ones vanishing in this particular setup. For example, $\Gamma^t_{tt} = 0$. That is because the metric will be stationary, given that the source $T_{\mu\nu}$ is constant. Further computations give

$$\Gamma^t_{ti} = \Gamma^i_{tt} = -\frac{1}{2}\partial_i h_{tt} = -\partial_i \bar{h}_{tt} \equiv -\vec{G}_i \tag{15}$$

$$\Gamma^i_{tj} = \frac{1}{2}\left(\partial_j h_{it} - \partial_i h_{jt}\right) \equiv -\frac{1}{2}\varepsilon_{ijk}\vec{H}_k \tag{16}$$

where we used that, for example, h_{ii} is subleading to h_{tt} for small velocities (coming from equation (12)).

All other symbols only contribute at order $O(v^2)$ to the geodesic equation. The only remaining relevant terms result in

$$\frac{dv^i}{dt} = -\Gamma^i_{tt} - 2\Gamma^i_{jt} v^j = \vec{G}_i + (v \times \vec{H})_i\,. \tag{17}$$

The definitions of G and H were chosen to match the form of the Lorentz force.

(e) We want to solve the equations

$$\begin{aligned}
\nabla^2 A_0(\mathbf{x}) &= 4\frac{M}{R^2}\delta(r-R)\,,\\
\nabla^2 A_1(\mathbf{x}) &= -4\frac{M}{R}\Omega\sin\theta\sin\phi\;\delta(r-R)\,,\\
\nabla^2 A_2(\mathbf{x}) &= 4\frac{M}{R}\Omega\sin\theta\cos\phi\;\delta(r-R)\,.
\end{aligned} \tag{18}$$

where we used the fact that the system is stationary to say that the time derivatives will vanish. For the time component of A, this is the usual equation for the electric potential of a charged spherical conductor, with the solution

$$A_0(\mathbf{x}) = \begin{cases} -\frac{M}{4\pi r} & r > R\,,\\ -\frac{M}{4\pi R} & 0 \le r \le R\,. \end{cases} \tag{19}$$

where the correct boundary condition have been imposed. The gravito-electric field is thus

$$\vec{G}(\mathbf{x}) = \begin{cases} \frac{M}{4\pi r^2} & r > R\,,\\ 0 & 0 \le r \le R\,. \end{cases} \tag{20}$$

For the other two components, we solve the associated Poisson equation with the Green's function

$$A_i(\mathbf{x}) = \frac{1}{4\pi}\int d^3\mathbf{x}'\,\frac{J_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{21}$$

In our case,

$$\begin{aligned}A_1(\mathbf{x}) &= \frac{M}{\pi R}\Omega \int r^2 \sin\theta dr d\theta d\phi \frac{\sin\theta \sin\phi \delta(r-R)}{|\mathbf{x}-\mathbf{x}'|} \\ A_2(\mathbf{x}) &= -\frac{M}{\pi R}\Omega \int r^2 \sin\theta dr d\theta d\phi \frac{\sin\theta \sin\phi \delta(r-R)}{|\mathbf{x}-\mathbf{x}'|}\end{aligned} \tag{22}$$

In spherical polar coordinates, we have that

$$\begin{aligned}|\mathbf{x}-\mathbf{x}'| &= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \\ &= \sqrt{r^2+r'^2-2rr'[\sin\theta\sin\theta'\cos(\phi-\phi')+\cos\theta\cos\theta']}\,.\end{aligned} \tag{23}$$

This problem appears in example 5.10 of Griffiths and 5.13 (or 5.6 depending on the edition) of Jackson. They propose two solutions: one is to go to a reference frame where $\mathbf{x}$ is aligned with the z axis rather than $\vec{\Omega}$ being aligned with it. This makes the integral much simpler. The second is to decompose $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in spherical harmonics. We invite you to look at those solutions. The answer is, inside the shell,

$$\begin{aligned}A_1(\mathbf{x}) &= -\frac{4M\Omega}{3R}y\,, \\ A_2(\mathbf{x}) &= \frac{4M\Omega}{3R}x\,.\end{aligned} \tag{24}$$

This corresponds to a uniform gravito-magnetic field

$$\vec{H} = -\frac{8M\Omega}{3R}\hat{z}\,. \tag{25}$$

(f) Let's take a stationary observer, so $U = \partial_t$. Let's observe what happens with a vector V along his trajectory (or equivalently, how observer would see it changing in time). V obeys parallel transport equation $U^\mu \nabla_\mu V^\nu = 0$:

$$U^\mu \nabla_\mu V^\nu = \nabla_t V^\nu = \partial_t V^\nu + \Gamma^\nu_{t\mu} V^\mu \tag{26}$$

Let's start with t component:

$$\partial_t V^t = -\Gamma^t_{t\mu} V^\mu = -\Gamma^t_{tj} V^j = \vec{G}\cdot\vec{V} \tag{27}$$

As in this setup $\vec{G} = 0$, nothing exciting happens. For spatial components the story is different:

$$\partial_t V^i = -\Gamma^i_{tt} V^t - \Gamma^i_{tj} V^j = \vec{G}_i V^t + \frac{1}{2}(\vec{V}\times\vec{H})_i = \frac{1}{2}(\vec{V}\times\vec{H})_i \tag{28}$$

Plugging in $\vec{H} = \nabla\times\vec{A} = -\frac{8M\Omega}{3R}\hat{z}$

$$\partial_t V^x = \frac{4M\Omega}{3R}V^y \tag{29}$$

$$\partial_t V^y = -\frac{4M\Omega}{3R}V^x \tag{30}$$

$$\partial_t V^z = 0 \tag{31}$$

which describes precession with a period of $\frac{3\pi R}{2M\Omega}$.

3. Gravitational Plane Waves

(a) Consider the spatial components of the Geodesic equation. If the particles start at rest, $\frac{dx^i}{d\tau} = 0$ and what remains is

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = -\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2\Big|_{\tau=0}. \tag{32}$$

The Christoffel symbol is given, at leading order in h, by

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\lambda}\left(2\partial_0 h^{\rm TT}_{0\lambda} - \partial_\lambda h^{\rm TT}_{00}\right) = 0 \tag{33}$$

where we used the gauge condition to say that the terms in the brackets are all zero. That means a particle at rest feels no acceleration from the passage of a gravitational wave.

(b) Let us consider slowly moving particles. Then, the four-velocity field is $U^\mu = (1,0,0,0)$ plus corrections of order $h^{\rm TT}_{\mu\nu}$. But we know that $R_{\mu\nu\rho\sigma}$ is already first order in h, so in the geodesic deviation formula we can really keep $U^\mu = (1,0,0,0)$. Then, the relevant entries of the Riemann tensor are

$$R_{\mu 00\sigma} = \frac{1}{2}(\partial_0^2 h^{\rm TT}_{\mu\sigma} + \partial_\sigma\partial_\mu h^{\rm TT}_{00} - \partial_\sigma\partial_0 h^{\rm TT}_{\mu 0} - \partial_\mu\partial_0 h^{\rm TT}_{\sigma 0}) = \frac{1}{2}\partial_0^2 h^{\rm TT}_{\mu\sigma}. \tag{34}$$

(c) Slowly moving particles are characterized by $\tau \sim t$ and all the Christoffel symbols vanish for any τ along the geodesic of the particle around which we are building the reference frame. The geodesic deviation equation can thus be written as

$$\frac{d^2S^\mu}{dt^2} = \frac{1}{2}S^\sigma\partial_t^2 h^{{\rm TT}\mu}_{\sigma}. \tag{35}$$

Now we specialize to the right handed wave,

$$h^{\rm TT}_{\mu\nu} = h_R\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & -i & 0\\ 0 & -i & -1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}e^{ik_\sigma x^\sigma}. \tag{36}$$

The two components of S^μ that will be affected by the wave are thus

$$\begin{cases}\frac{d^2S^1}{dt^2} &= -\frac{\omega^2}{2}h_R{\rm Re}\left[(S^1 - iS^2)e^{-i\omega t}\right]\\ \frac{d^2S^2}{dt^2} &= -\frac{\omega^2}{2}h_R{\rm Re}\left[(-iS^1 - S^2)e^{-i\omega t}\right]\end{cases} \tag{37}$$

where we've taken $k^\sigma = (\omega,0,0,\omega)$ and $x^\sigma = (t,x,y,0)$. Taking the real part and solving this equation with Mathematica (equivalently, using the ansatz provided in the Problem Set), we get

$$\begin{cases}S^1(t) = \frac{1}{2}\left[S_0^1(1+\cos(\omega t)) - S_0^2\sin(\omega t) + (2v_0^1 + S_0^2\omega)\,t\right]\\ S^2(t) = \frac{1}{2}\left[S_0^2(3-\cos(\omega t)) - S_0^1\sin(\omega t) + (2v_0^2 + S_0^1\omega)\,t\right]\end{cases} \tag{38}$$

where S_0^i are the initial values of $S^i(t)$ and v_0^i are the initial values of $\frac{d}{dt}S^i(t)$. From the solution we can see that, for fine tuned initial conditions such that the linear term in t vanishes, we get circles in the $\{S^1(t), S^2(t)\}$ plane, as expected from a circularly polarized wave. The linear term is an artifact due to the unphysical situation we are studying: this is an eternal gravitational wave, and there is really no frame of reference at any point in which a set of particles traversed by a gravitational wave is all at rest at the same time. A more careful treatment would be to turn on the gravitational wave adiabatically and see how the set of particles behaves. On Moodle you will find animations made from these equations for all polarizations h_R, h_L, h_+ and $h_\times$, where we fine tuned the initial condition of each particle such that the linear term vanishes.