Relativity and Cosmology I

Exam Solutions - January 2024

1. Time delays

(a) The wordlines of the two clocks can be parametrized as

$$\begin{aligned} x_1^\mu(t) &= (t, R_1, \theta_1, \phi_1)\,, \\ x_2^\mu(t) &= (t, R_2, \theta_2, \phi_2)\,, \end{aligned} \tag{1}$$

where $\{R_i, \theta_i, \phi_i\}$ are constants. The proper time element is

$$d\tau^2 = (1+2\Phi)\mathrm{d}t^2 - (1-2\Phi)\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,, \qquad \Phi = -\frac{GM}{r}\,. \tag{2}$$

The proper time along each worldline as a function of time is

$$\begin{aligned} \tau_1(t) &= \int_0^t dt\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = \int_0^t dt\sqrt{1-\frac{2GM}{R_1}} = t\sqrt{1-\frac{2GM}{R_1}}\,, \\ \tau_2(t) &= t\sqrt{1-\frac{2GM}{R_2}}\,. \end{aligned} \tag{3}$$

The amounts of ticks a clock can make in a certain interval of coordinate time Δt is proportional to the associated proper time interval. Since $R_1 < R_2$, we see that $\tau_2 > \tau_1$, so the clock on the tower ticks more in the same amount of coordinate time relative to the clock on the ground. The clock on the tower thus ticks faster.

Extra comments:

Notice that we neglected the velocity due to the rotation of the Earth. Taking this into account, we would find

$$\tau_i(t) = t\sqrt{1-\frac{2GM}{R_i} - R_i^2\Omega^2\sin^2\theta}\,. \tag{4}$$

where $\Omega = \frac{d\phi}{dt}$ is the angular velocity of the Earth and θ is a constant depending on the position of the clocks on the Earth. For the Earth, we have

$$10^{-9} \sim \frac{GM}{c^2R_1} \gg \frac{R_1^2\Omega^2}{c^2} \sim 10^{-12}\,. \tag{5}$$

The careful student may also worry that the rotation of the Earth should produce a non-spherically symmetric metric (similar to the Kerr metric). These effects are also small. We can estimate them noticing that the Kerr metric differs from the

Schwarzschild metric (in relative terms) by order $a^2/(r_S r)$ where $r_S = 2GM$ and $a = J/M$, with J the angular momentum. But $r \geq R_1$ outside the Earth and $J \sim MR_1^2\Omega$. Therefore, the rotation of the Earth leads to a correction of order

$$\frac{a^2}{r_S r} \sim \frac{R_1^3\Omega^2}{GM} \sim 10^{-3}\,, \tag{6}$$

relative to the non-rotating approximation.

(b) We are in the Newtonian approximation. The worldline of the clock in orbit can be parametrized as

$$x_3^\mu(t) = (t, R_3, \pi/2, \Omega t)\,. \tag{7}$$

The angular velocity can be related to the radial position R_3 through the centripetal acceleration

$$m\Omega^2 R_3 = G\frac{mM}{R_3^2} \to \Omega^2 = \frac{MG}{R_3^3}\,. \tag{8}$$

Then, its proper time is

$$\tau_3(t) = \int_0^t dt\sqrt{1 - \frac{2GM}{R_3} - R_3^2\Omega^2} = t\sqrt{1 - \frac{3GM}{R_3}}\,. \tag{9}$$

(c) The two clocks stay synchronized if their proper times match

$$\tau_1(t) = \tau_3(t)\,. \tag{10}$$

This is only possible if

$$R_3 = \frac{3}{2}R_1\,. \tag{11}$$

(d) The time difference between a GPS satellite and a clock on Earth is given by

$$\tau_3(t) - \tau_1(t) = t\left(\sqrt{1 - \frac{3GM}{c^2R_3}} - \sqrt{1 - \frac{2GM}{c^2R_1}}\right) \approx \frac{GMt}{c^2}\left(\frac{1}{R_1} - \frac{3}{2R_3}\right)\,. \tag{12}$$

Substituting $t = 1$ day, $R_3 = 2\times10^7 m + R_1$ and the rest of the constants, we obtain

$$\tau_3(1\text{day}) - \tau_1(1\text{day}) \approx 36\mu\text{s} \tag{13}$$

2. Period change

(a) As in problem 1, we start by relating the angular velocity to the radius of the orbit through the centripetal acceleration, obtaining again

$$\Omega = \sqrt{\frac{MG}{R^3}}\,. \tag{14}$$

Angular velocity and period are related through $T = \frac{2\pi}{\Omega}$, giving

$$R = \left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}\,. \tag{15}$$

The total mechanical energy is given by the kinetic and the potential energies

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{1}{2}\frac{GMm}{R} = -m\left(\frac{\pi GM}{\sqrt{2}T}\right)^{\frac{2}{3}}\,. \tag{16}$$

(b) In the current approximation, the star is fixed and the planet orbits around it. The energy density is thus

$$T^{00}(t,\vec{y}) = m\delta(x - R\cos\Omega t)\delta(y - R\sin\Omega t)\delta(z) + M\delta(x)\delta(y)\delta(z)\,. \tag{17}$$

The quadrupole moment is then

$$I_{ij}(t) = \int y^i y^j T^{00}(t,\vec{y}) d^3y\,, \tag{18}$$

with non-vanishing components

$$I_{xx} = mR^2\cos^2\Omega t\,, \quad I_{yy} = mR^2\sin^2\Omega t\,, \quad I_{xy} = mR^2\cos\Omega t\sin\Omega t = I_{yx}\,. \tag{19}$$

(c) To compute the average power emitted we first compute the traceless part of the quadrupole

$$J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}I_{kl}\,, \tag{20}$$

with non-vanishing components

$$\begin{aligned} J_{xx} &= mR^2\left(\cos^2\Omega t - \frac{1}{3}\right), \qquad & J_{yy} &= mR^2\left(\sin^2\Omega t - \frac{1}{3}\right), \\ J_{xy} &= mR^2\sin\Omega t\cos\Omega t = J_{yx}, \qquad & J_{zz} &= -\frac{1}{3}mR^2. \end{aligned} \tag{21}$$

The power emitted is given by

$$P = -\frac{G}{5}\left\langle \frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}\right\rangle \tag{22}$$

The non zero third time derivatives of J are

$$\begin{aligned} \dddot{J}_{xx} &= 4\Omega^3 mR^2\sin(2\Omega t)\,, \qquad & \dddot{J}_{yy} &= -4\Omega^3 mR^2\sin(2\Omega t)\,, \\ \dddot{J}_{xy} &= -4\Omega^3 mR^2\cos(2\Omega t) = \dddot{J}_{yx}\,. \end{aligned} \tag{23}$$

The average power is thus

$$P = -\frac{G}{5}32\Omega^6 m^2R^4\left(\sin^2(2\Omega t) + \cos^2(2\Omega t)\right) = -\frac{32}{5}G\Omega^6 m^2R^4\,. \tag{24}$$

In terms of the period, we obtain

$$P = -\frac{256}{5}m^2\left(\frac{2\pi^{10}G^7M^4}{T^{10}}\right)^{\frac{1}{3}} \tag{25}$$

(d) Let us start from the relation between energy and period

$$E = -m\left(\frac{\pi GM}{\sqrt{2}T}\right)^{\frac{2}{3}}\,. \tag{26}$$

We differentiate both sides with respect to time

$$P = \frac{m}{3}\left(2\pi GM\right)^{2/3}\frac{1}{T^{5/3}}\frac{dT}{dt} \tag{27}$$

we substitute (25) into (27), and obtain

$$T^{5/3}dT = -\frac{384}{5}m\left(4M^2\pi^8G^5\right)^{1/3}dt \tag{28}$$

Integrating, we obtain

$$T(t) = \left(T(0)^{8/3} - \frac{1024}{5}m(4M^2\pi^8G^5)^{1/3}t\right)^{3/8}. \tag{29}$$

(e) The orbit of the Earth around the sun today lasts one year, $T(0) = 1$ yr. The change in this period is given by

$$\begin{aligned} T(0) - T(t) =& T(0)\left[1 - \left(1 - \frac{1024}{5}m\left(\frac{4M^2\pi^8G^5}{T(0)^8}\right)^{1/3}t\right)^{3/8}\right] \\ \approx& \frac{384}{5}m\left(\frac{4M^2\pi^8G^5}{T(0)^5}\right)^{1/3}t \end{aligned} \tag{30}$$

Imposing the left hand side to be 1 second, we obtain

$$t = \frac{5}{384}\left(\frac{T(0)^5c^{15}}{4\pi^8m^3M^2G^5}\right)^{1/3}(1s) \approx 9.1\times 10^{15}\,\text{years}, \tag{31}$$

the phenomenon can thus be largely neglected.

(f) In fact, the planet and the star both orbit around the center of mass (that can be put at the origin without loss of generality) with respective radii

$$R = \frac{M}{M+m}\tilde{r}, \qquad r = \frac{m}{M+m}\tilde{r}, \tag{32}$$

where $\tilde{r} = R + r$ is the distance between the star and the planet. Thanks to the symmetry of the problem, to estimate the power emitted by the star it suffices to switch $R \to r = \frac{m}{M}R$ and $m \to M$ in (23) and add it to the traceless part of the quadrupole J of the planet. Then, we obtain the total power

$$P_{\text{tot}} = -\frac{32}{5}G\Omega^6m^2R^4\left(1 + O\left(\frac{m}{M}\right)\right). \tag{33}$$

Notice that the correction is negligible for the system Earth-Sun.

3. Black hole shadow

(a) First of all, recall that (affinely parametrized) geodesics can be found as the equations of motion of the Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu, \tag{34}$$

where $\dot{x}^\mu \equiv dx^\mu/d\lambda$, with λ an affine parameter along the null geodesic.

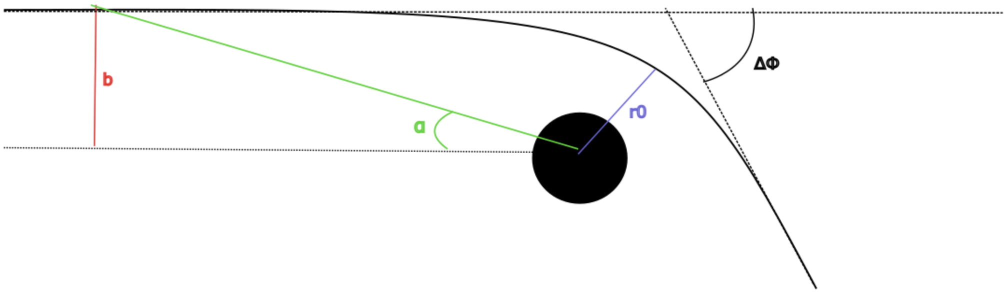

Figure 1: Scheme of a photon scattering on a black hole, with b the impact parameter, r_0 the distance of closest approach, $\Delta\phi$ the deflection angle and α is the angle between the (spatial) position vector $\vec{r}$ and the (spatial) momentum $\vec{p}$ of the photon.

We are interested in motion in the plane $\theta = \pi/2$ of the Schwartzschild geometry for which we get

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = -\left(1 - \frac{r_S}{r}\right)\dot{t}^2 + \frac{1}{1 - \frac{r_S}{r}}\dot{r}^2 + r^2\dot{\phi}^2. \tag{35}$$

Since the variables t and ϕ are cyclic (they do not appear in the Lagrangian) we get two obvious conserved quantities

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{t}} &= \text{const} \implies \left(1 - \frac{r_S}{r}\right)\dot{t} \equiv E = \text{const}, \\ \frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \text{const} \implies r^2\dot{\phi} \equiv L = \text{const}. \end{aligned} \tag{36}$$

A physical interpretation of those conserved quantities comes from looking in the asymptotic region $r \to \infty$ where the geometry becomes Minkowski spacetime and $E = \dot{t} = p^0$ is the energy of the particle while $L = r^2\dot{\phi} = p_\phi$ is its angular momentum.

A further constraint on null geodesics is that they have zero length, hence

$$0 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \implies -\frac{E^2}{1 - \frac{r_S}{r}} + \frac{\dot{r}^2}{1 - \frac{r_S}{r}} + \frac{L^2}{r^2} = 0, \tag{37}$$

where we used the conserved charges to replace $\dot{t}$ and $\dot{\phi}$. Now we can use the chain rule

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \frac{dr}{d\phi}\frac{L}{r^2}. \tag{38}$$

Plugging this result in (37) we get

$$\left(\frac{dr}{d\phi}\right)^2 - r^4\frac{E^2}{L^2} + r^2\left(1 - \frac{r_S}{r}\right) = 0, \tag{39}$$

which identifies

$$V(r) = r^2\left(1 - \frac{r_S}{r} - r^2\frac{E^2}{L^2}\right). \tag{40}$$

(b) As explained before, in the asymptotic region the conserved quantities E and L are respectively the energy and angular momentum of the photon. The angular momentum can also be computed as

$$\vec{L} = \vec{r} \wedge \vec{p} = rp \sin \alpha \vec{e}_z. \tag{41}$$

The spatial momentum of the photon p is simply related to its energy by $p = E$, and usual trigonometric relations, see fig. 1, yield $b = r \sin \alpha$, where b is the impact parameter. Therefore we get

$$L = bE \Leftrightarrow b = \frac{L}{E}. \tag{42}$$

The equation of motion can then be written

$$\left(\frac{dr}{d\phi}\right)^2 + V(r) = 0, \qquad V(r) = r^2 \left(1 - \frac{r_S}{r} - \frac{r^2}{b^2}\right). \tag{43}$$

Another way to find that the impact parameter is $b = L/E$ is by considering the infinite radius limit in (39). As $r \to \infty$, we can neglect the term r_S/r in (39). Then, we can check that the straight line trajectory

$$b = r \sin \phi\,, \tag{44}$$

is an exact solution of (39) with $r_S = 0$ if $b = L/E$.

(c) The equation of motion (43) is a one dimensional particle of zero energy moving in the potential $V(r)$, shown on fig. 2. If there is a region with $V > 0$ between $r = \infty$ and $r = r_S$, the potential barrier is too high and the particle does not have enough energy to go through. Therefore let us compute the maxima of the potential. We

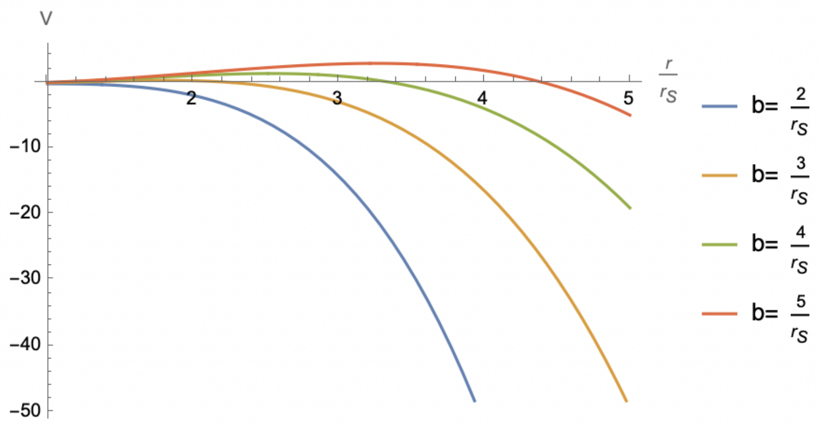

Figure 2: Potential energy of the photon for different impact parameters b.

have

$$V'(r) = -r_S + 2r - \frac{4r^3}{b^2}. \tag{45}$$

Therefore the position of the maxima r_* satisfies

$$-r_S + 2r_* - \frac{4r_*^3}{b^2} = 0 \Leftrightarrow \frac{r_*^3}{b^2} = \frac{1}{4}(2r_* - r_S). \tag{46}$$

The critical impact parameter b_c will be the one for which $V(r_*) = 0$. Therefore we look for solutions of

$$r_* - r_S - \frac{r_*^3}{b^2} = 0 = r_* - r_S - \left(\frac{r_*}{2} - \frac{r_S}{4}\right) = \frac{r_*}{2} - \frac{3}{4} r_S. \tag{47}$$

Therefore $r_*(b)$ satisfies

$$r_*(b_c) = \frac{3}{2} r_S. \tag{48}$$

Plugging this constraint in (46) we get

$$2r_S - \frac{27 r_S^3}{2b_c^2} = 0 \implies b_c = \frac{3\sqrt{3}}{2} r_S. \tag{49}$$

(d) Finally we want to express the function $\phi(r)$ and compute the impact parameter such that

$$\Delta\Phi = 3\pi. \tag{50}$$

From (43) we have

$$d\phi = -\frac{dr}{r} \frac{1}{\sqrt{r^2/b^2 + r_S/r - 1}}, \tag{51}$$

where we chose the minus sign since r decreases when ϕ increases. The total angle $\Delta\phi$ is twice the angle between $r = \infty$ and $r = r_0$ where r_0 is the point of closest approach, that solves $dr/d\phi = 0$ which together with (43) gives $V(r_0) = 0$. We therefore get the relation

$$1 - \frac{r_S}{r_0} - \frac{r_0^2}{b^2} = 0 \implies b = \frac{r_0}{\sqrt{1 - r_S/r_0}}. \tag{52}$$

Therefore we want to solve

$$\frac{3}{2}\Delta\phi = \frac{3\pi}{2} = \int_{r_0}^{\infty} \frac{dr}{r} \frac{1}{\sqrt{r^2/b^2 + r_S/r - 1}}. \tag{53}$$

There is no analytical solution but we can give it to a computer. However computers take numbers as inputs, and not physical parameters like r_S or b. Therefore we start by writing the integral in dimensionless quantities by defining $\rho = r_0 r$, yielding

$$\frac{3\pi}{2} = \int_1^{\infty} \frac{d\rho}{\rho} \frac{1}{\sqrt{\rho^2 \frac{r_0^2}{b^2} + \frac{r_S}{r_0}\frac{1}{\rho} - 1}}. \tag{54}$$

Using (52) for the first term in the square root we get

$$\frac{3\pi}{2} = \int_1^{\infty} \frac{d\rho}{\rho} \frac{1}{\sqrt{\rho^2 - 1 + \frac{r_S}{r_0}\left(\frac{1}{\rho} - \rho^2\right)}} \equiv F(r_S/r_0). \tag{55}$$

The function $F(x)$ can be plotted by a computer. We then just have to find for which value it intersects $\frac{3\pi}{2}$ which will give the point of closest approach r_0 for which the photon does exactly one rotation around the black hole before escaping. Then the relation (52) relates this r_0 to the impact parameter b_1. We get

$$r_S/r_0 = 0.644... \implies b_1 = (2.602...) \times r_S. \tag{56}$$

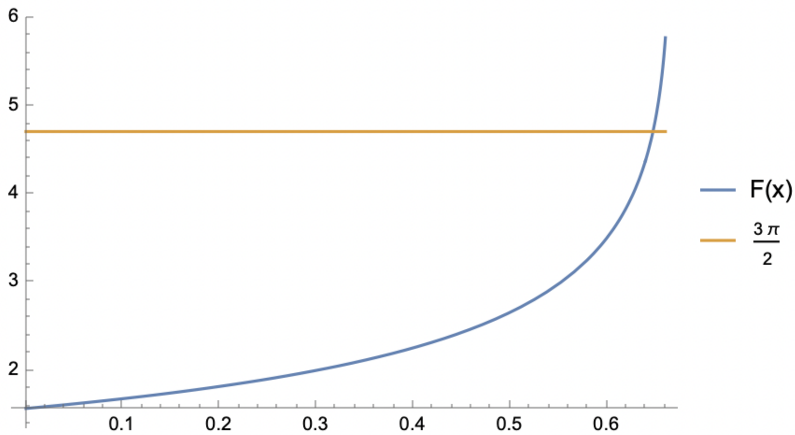

Figure 3: Intersection of the solution of the equations of motion $\phi(r_0)$ with $\frac{3\pi}{2}$.