QUANTUM PHYSICS III

Solutions to Problem Set 6 15 October 2024

1. Symmetry restoration in the double-well potential

1. Let $\psi_0(x)$ be the low energy bound state wave function of the left well. Then, one can build two eigenfunctions of the whole double-well potential as follows,

$$\psi_1(x) = \frac{1}{\sqrt{2}}(\psi_0(x) + \psi_0(-x)) \ , \quad \psi_2(x) = \frac{1}{\sqrt{2}}(\psi_0(x) - \psi_0(-x)) \ . \tag{1}$$

For the initial wave function of the particle in the left well we have

$$\Phi(x, 0) = \psi_0(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x)) \ . \tag{2}$$

Time evolving this wave packet, one finds

$$\Phi(x, t) = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}E_1 t}(\psi_1(x) + e^{-i\delta t}\psi_2(x)) \ , \quad \delta = \frac{E_2 - E_1}{\hbar} > 0 \ , \tag{3}$$

where δ is the energy splitting between the states represented by $\psi_1(x)$ and $\psi_2(x)$. The probability to detect the particle at the position x at the time t is then given by

$$P(x, t) = |\Phi\Phi^*| = \frac{1}{2}(\psi_1(x)^2 + \psi_2(x)^2 + 2\psi_1(x)\psi_2(x)\cos\delta t) \ . \tag{4}$$

2. The probability for the particle to be found in the right well at the time t can be written as

$$P(t) = \int_0^\infty P(x, t)\, dx \ . \tag{5}$$

We substitute eq. (4) into eq. (5), and note that integration of any term including the function $\psi_0(x)$ gives zero, since this function is localized in the left well, and that

$$\int_0^\infty \psi_0(-x)^2 dx = 1 \ , \tag{6}$$

because of the normalization of the bound state wave function. Hence,

$$P(t) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} - 2\cos\delta t \cdot \frac{1}{2}\right) = \sin^2\frac{\delta t}{2} \ . \tag{7}$$

It remains to compute exactly the energy splitting δ,

$$\delta = \frac{\hbar\omega}{\pi}\exp\left(-\frac{1}{\hbar}\int_{-x^*}^{x^*} |p| dx\right) \ , \tag{8}$$

where $\pm x^*$ are the turning points of the subbarrier transition, and ω is the frequency of the classical oscillations in the well. The expression (8) can be easily computed in the limit $E \ll V_0$. Indeed, near the left well bottom the potential is well approximated by a parabolic function,

$$V(y) = V_0 y^2 (y + 2x_1)^2 \approx 4x_1^2 V_0 y^2 \ , \quad y = x - x_1 \ , \quad |y/x_1| \ll 1 \ . \tag{9}$$

From here, the oscillation frequency is extracted as

$$V(y) = \frac{1}{2} m\omega^2 y^2 \quad \Rightarrow \quad \omega^2 = \frac{8x_1^2 V_0}{m} \ . \tag{10}$$

Next, the integral in (8) is evaluated as follows,

$$\int_{-x^*}^{x^*} |p| dx \approx \sqrt{2mV_0} \int_{-x_1}^{x_1} \sqrt{(x - x_1)^2 (x + x_1)^2} dx = \sqrt{2mV_0} \cdot \frac{4}{3} x_1^3 \ . \tag{11}$$

Thus,

$$\delta = \sqrt{\frac{8\hbar^2 x_1^2 V_0}{m\pi^2}} \exp\left(-\frac{4}{3\hbar} \sqrt{2mV_0} x_1^3 \right) \ . \tag{12}$$

3. The average probability to detect the particle in the right well during the time T is given by

$$\frac{1}{T} \int_0^T P(t)\, dt \ . \tag{13}$$

Taking the limit $T \to \infty$, we obtain

$$\lim_{T\to\infty} \frac{1}{2T} \int_0^T (1 - \cos \delta t)\, dt = \frac{1}{2} - \frac{1}{2\delta} \lim_{T\to\infty} \frac{\sin \delta T}{T} = \frac{1}{2} \ . \tag{14}$$

So, indeed, we have the equal chance to find the particle in either well. In other words, in the large time limit, the system forgets its initial state and exhibits a universal behaviour. In particlar, the parity symmetry of the potential, broken by the initial distribution of the wave function, gets restored as the time passes by. Basically, this is the reason why in $1D$ quantum systems it is impossible to make a spontaneous symmetry breaking.

2. WKB spectrum of the Hydrogen atom

1. It will be convenient to use electron's momentum k related to its energy E as (we work in natural units $\hbar = c = 1$)

$$E = -\frac{k^2}{2M} \ . \tag{15}$$

Then, for the potential

$$V(r) = -\frac{1}{a_0 M r} + \frac{(l + 1/2)^2}{2Mr^2} \ , \tag{16}$$

we have

$$\int_{r_1}^{r_2} p\,dr = \int_{r_1}^{r_2} \sqrt{-k^2 + \frac{2}{a_0 r} - \frac{(l+1/2)^2}{r^2}}\,dr$$
$$= k\int_{r_1}^{r_2} \sqrt{\left(1-\frac{r_1}{r}\right)\left(\frac{r_2}{r}-1\right)}\,dr = \frac{k\pi}{2}(r_1 + r_2 - 2\sqrt{r_1 r_2})\,, \tag{17}$$

where

$$r_{1,2} = \frac{1}{a_0 k^2} \pm \sqrt{\frac{1}{(a_0 k^2)^2} - \frac{(l+1/2)^2}{k^2}} \tag{18}$$

are the turning points. Applying the Bohr-Sommerfeld quantization rule, we find

$$\pi\left(n_r + \frac{1}{2}\right) = \pi\left(\frac{1}{a_0 k} - l - \frac{1}{2}\right)\,, \tag{19}$$

or, using eq. (15),

$$E_{n_r} = -\frac{1}{2Ma_0^2}\frac{1}{(n_r + l + 1)^2}\,. \tag{20}$$

This coincides with the exact answer.

2. The ground state level is given by $n_r = l = 0$, and we have

$$E_0 = -\frac{1}{2Ma_0^2}\,. \tag{21}$$

In natural units, the Bohr radius equals $a_0 = 2.68 \cdot 10^{-4} eV^{-1}$, while the electron mass $M = 5.11 \cdot 10^5 eV$, hence

$$E_0 = -13.6\,eV\,. \tag{22}$$

3. From eq. (20) we see that the energy of the level depends on the sum $n_r + l$, and it can be the same for different values of the radial number n_r. Therefore, the energy levels are degenerate. One can rewrite eq. (20) as

$$E_n = -\frac{1}{2Ma_0^2}\frac{1}{n^2}\,, \tag{23}$$

where $n = n_r + l + 1$ is called a principal quantum number. It can take values $n = 1, 2, \ldots$. For any fixed n, there are $n - 1$ possible values of the orbital momentum l, and for any fixed l, there are $2l+1$ possible values of the magnetic number m. Recall also that the levels are additionally degenerate due to the electron spin $s = \pm\frac{1}{2}$. So, the full degeneracy of the n'th energy level is

$$2\sum_{l=0}^{n-1}(2l+1) = 2n^2\,. \tag{24}$$

3. Classical scattering on a Coulomb potential

1. The energy of the particle moving in the potential $U(r)$ in two dimensions is written in polar coordinates as follows,

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + U(r)\,, \tag{25}$$

where $L = mr^2\dot{\phi}$ is the angular momentum. Expressing $\dot{r}$ from the relation above, we have

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}(E - U(r)) - \frac{L^2}{m^2r^2}}\,. \tag{26}$$

Next, we observe that as long as $\dot{r} \neq 0$,

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{dt}{dr} = \frac{L}{mr^2}\frac{1}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{L^2}{m^2r^2}}}\,. \tag{27}$$

Taking the integral, we obtain

$$\phi(r) = \int_{\infty}^{r} \frac{L/r^2 dr}{\sqrt{2m(E - U(r)) - L^2/r^2}}\,. \tag{28}$$

2. The deflection angle θ is given by (see figure 1)

$$\theta = |\pi - 2\phi_0|\,, \tag{29}$$

where ϕ_0 is the angle between the direction to the minimum distance from the scattering center to particle's orbit and the direction to the infinite distance between them. Using eq. (28) with $U(r) = \alpha/r$, we write

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{Ldr}{r\sqrt{2mEr^2 - 2m\alpha r - L^2}}\,. \tag{30}$$

This integral can be taken analytically, the answer is

$$\phi_0 = \arcsin \frac{-2m\alpha r - 2L^2}{r\sqrt{4m^2\alpha^2 + 8L^2mE}}\Bigg|_{r_{min}}^{\infty}\,. \tag{31}$$

It remains to find r_{min}. It is the point at which $\dot{r} = 0$. From eq. (26) it then follows that

$$r_{min} = \frac{\alpha}{2E} + \frac{1}{2E}\sqrt{\alpha^2 + 2L^2E/m}\,. \tag{32}$$

Substituting this into eq. (31), we have

$$\phi_0 = -\arcsin \frac{\alpha}{\sqrt{\alpha^2 + 2L^2E/m}} + \frac{\pi}{2}\,. \tag{33}$$

Hence,

$$\theta = 2\arcsin \frac{|\alpha|}{\sqrt{\alpha^2 + 2L^2E/m}}\,. \tag{34}$$

3. Since the energy and the angular momentum are conserved, one can write $E = mv_\infty^2/2$, hence $v_\infty = \sqrt{2E/m}$ and $L = mv_\infty b = \sqrt{2mE}b$. Eq.(34) is rewritten as

$$\theta = 2\arcsin\frac{|\alpha|}{\sqrt{\alpha^2 + 4E^2b^2}}\,. \tag{35}$$

4. Firstly, let us express b through θ :

$$b = -\frac{\alpha}{2E}\cot\frac{\theta}{2}\,. \tag{36}$$

Then, we note that $dN = 2\pi nbdb = 2\pi nb\frac{db}{d\theta}d\theta$, and $d\sigma = dN/n = 2\pi b\frac{db}{d\theta}d\theta$. On the other side, from eq. (36) we have

$$\frac{db}{d\theta} = \frac{\alpha}{2E}\frac{1}{2}\frac{1}{\sin^2(\theta/2)}\,. \tag{37}$$

Hence,

$$d\sigma = 2\pi\frac{\alpha}{2E}\cot\left(\frac{\theta}{2}\right)\frac{\alpha}{2E}\frac{1}{2}\frac{d\theta}{\sin^2(\theta/2)} = \pi\frac{\alpha^2}{4E^2}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}d\theta\,. \tag{38}$$

Finally, $d\Omega = 2\pi\sin\theta d\theta = 4\pi\sin(\theta/2)\cos(\theta/2)d\theta$, and we arrive at

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\frac{1}{\sin^4(\theta/2)}\,. \tag{39}$$

5. The integral over eq. (39) is divergent. Therefore, the total cross section is infinite. The physical interpretation of this is that the potential affects the motion of the particle regardless its distance to the scattering center. This is a typical example of the so-called long-range force.

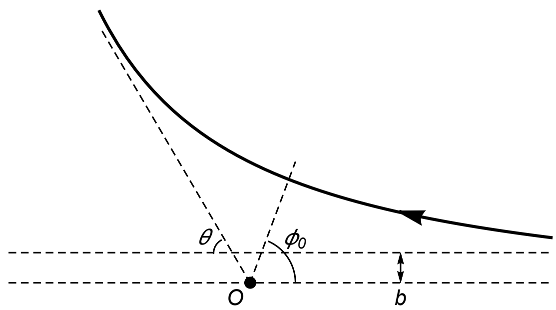

FIG. 1 – The scattering potential with $\alpha < 0$.

4. Differential cross section transformation

For the difference between the laboratory frame and the center-of-mass frame, see figure 2. In the center-of-mass frame, two particles are traveling towards each other. One particle with mass m_1 has a speed v_{C1} and is traveling in the $+x$-direction. The other particle with mass m_2 has a speed v_{C2} and is traveling in the $-x$-direction. Since we are in the center-of-mass frame, their momenta should be equal and opposite, so

$$v_{C2} = -\lambda v_{C1}\,, \tag{40}$$

where $\lambda = m_1/m_2$. After the collision, the first particle scatters into an angle θ_{CM} with the velocity u_{C1}. But $u_{C1} = v_{C1}$, because the collision is elastic. By the same reasoning, $u_{C2} = v_{C2}$. Expressing the velocities as vectors, we have

$$\begin{aligned} \vec{v}_{C1} &= v_{C1}\vec{i}\,, \\ \vec{u}_{C1} &= v_{C1}(\cos\theta_{CM}\vec{i} + \sin\theta_{CM}\vec{j})\,, \\ \vec{v}_{C2} &= -\lambda v_{C1}\vec{i}\,, \\ \vec{u}_{C2} &= \lambda v_{C1}(\cos\theta_{CM}\vec{i} + \sin\theta_{CM}\vec{j})\,. \end{aligned} \tag{41}$$

In the lab frame, the particle of mass m_1 comes in with the velocity v_{L1} and collides with the particle of mass m_2 with $v_{L2} = 0$ sending them both off in different directions. The scattered particle is deflected into an angle θ_{LAB} and has the velocity u_{L1}. The target particle is deflected into an angle θ_2 and has the velocity u_{L2}. As vectors, the velocities are

$$\begin{aligned} \vec{v}_{L1} &= v_{L1}\vec{i}\,, \\ \vec{u}_{L1} &= u_{L1}(\cos\theta_{LAB}\vec{i} + \sin\theta_{LAB}\vec{j})\,, \\ \vec{v}_{L2} &= 0\,, \\ \vec{u}_{L2} &= u_{L2}(\cos\theta_2\vec{i} - \sin\theta_2\vec{j})\,. \end{aligned} \tag{42}$$

To relate eqs. (41) and (42), we observe that the two reference frames are transformed to one another by a Galilean transformation, that is, to obtain the lab frame velocities, one should subtract v_{C2} in the x-direction from the center-of-mass frame velocities. Doing so, the velocities in the center-of-mass frame become,

$$\begin{aligned} \vec{v}_{L1} &= (1 + \lambda)v_{C1}\vec{i}\,, \\ \vec{u}_{L1} &= v_{C1}((\cos\theta_{CM} + \lambda)\vec{i} + \sin\theta_{CM}\vec{j})\,, \\ \vec{v}_{L2} &= 0\,, \\ \vec{u}_{L2} &= -\lambda v_{C1}((\cos\theta_{CM} - 1)\vec{i} + \sin\theta_{CM}\vec{j})\,. \end{aligned} \tag{43}$$

Comparing eqs. (42) and (43), one finds

$$\begin{aligned} u_{L1}\cos\theta_{LAB} &= v_{C1}(\cos\theta_{CM} + \lambda)\,, \\ u_{L1}\sin\theta_{LAB} &= v_{C1}\sin\theta_{CM}\,. \end{aligned} \tag{44}$$

Hence,

$$\tan\theta_{LAB} = \frac{\sin\theta_{CM}}{\cos\theta_{CM} + \lambda}\,, \tag{45}$$

or

$$\cos\theta_{LAB} = \frac{1}{\sqrt{1 + \tan^2\theta_{LAB}}} = \frac{\cos\theta_{CM} + \lambda}{\sqrt{1 + 2\lambda\cos\theta_{CM} + \lambda^2}}\,. \tag{46}$$

Since the total cross section should not depend on the reference frame, $d\sigma$ should be the same in either the lab frame or the center-of-mass frame. However, since there is an angular dependence in $d\Omega$, the differential cross section is different in different frames. Since

$$d\sigma = \frac{d\sigma}{d\Omega} d\Omega\,, \tag{47}$$

we know that

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} d\Omega_{CM} = \left.\frac{d\sigma}{d\Omega}\right|_{LAB} d\Omega_{LAB}\,. \tag{48}$$

Therefore,

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \left.\frac{d\sigma}{d\Omega}\right|_{LAB} \frac{d\Omega_{LAB}}{d\Omega_{CM}}\,. \tag{49}$$

Since $d\Omega = 2\pi \sin\theta d\theta$, we have,

$$\frac{d\Omega_{LAB}}{d\Omega_{CM}} = \frac{\sin\theta_{LAB} d\theta_{LAB}}{\sin\theta_{CM} d\theta_{CM}}\,. \tag{50}$$

Now, taking the derivative of eq. (46), we find

$$-\sin\theta_{LAB} d\theta_{LAB} = -\sin\theta_{CM} d\theta_{CM} \left(\frac{1+\lambda\cos\theta_{CM}}{(1+2\lambda\cos\theta_{CM}+\lambda^2)^{3/2}} \right)\,. \tag{51}$$

We then see by plugging this into eq. (50) that

$$\frac{d\Omega_{LAB}}{d\Omega_{CM}} = \frac{1+\lambda\cos\theta_{CM}}{(1+2\lambda\cos\theta_{CM}+\lambda^2)^{3/2}}\,, \tag{52}$$

and, finally,

$$\left.\frac{d\sigma}{d\Omega}\right|_{LAB} = \frac{(1+2\lambda\cos\theta_{CM}+\lambda^2)^{3/2}}{|1+\lambda\cos\theta_{CM}|} \left.\frac{d\sigma}{d\Omega}\right|_{CM}\,. \tag{53}$$

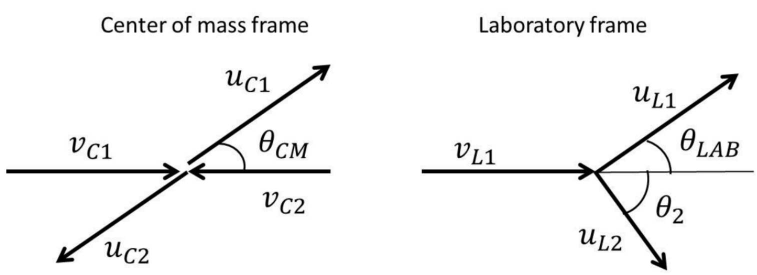

FIG. 2 – Different reference frames

5. Interaction picture

1. Recalling the relation between states and operators in the Schroedinger and Heisenberg pictures, we have

$$\begin{aligned} \Psi_I(t) &= \hat{U}_0^\dagger(t)\Psi_S(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\Psi_H\,, \\ \hat{A}_I(t) &= \hat{U}_0^\dagger(t)\hat{A}_S\hat{U}_0(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\hat{A}_H(t)\hat{U}^\dagger(t)\hat{U}_0(t)\,. \end{aligned} \tag{54}$$

2. The evolution equation for the wave function in the interaction picture is obtained straightforwardly :

$$
\begin{aligned}
-\frac{\hbar}{i}\frac{d}{dt}\Psi_I(t) &= -\frac{\hbar}{i}\frac{d}{dt}\hat{U}_0^\dagger(t)\Psi_S(t) = -\frac{\hbar}{i}\frac{d\hat{U}_0^\dagger(t)}{dt}\Psi_S(t) - \frac{\hbar}{i}\hat{U}_0^\dagger(t)\frac{d\Psi_S(t)}{dt} \\
&= -\hat{U}_0^\dagger(t)\hat{H}_0\Psi_S(t) + \hat{U}_0^\dagger(t)(\hat{H}_0 + \hat{V})\Psi_S(t) \\
&= \hat{U}_0^\dagger(t)\hat{V}\hat{U}_0(t)\Psi_I(t) = \hat{V}_I(t)\Psi_I(t) \ ,
\end{aligned}
\tag{55}
$$

where in the last line we used the fact that $\Psi_S(t) = \hat{U}_0(t)\Psi_I(t)$.

3. Similarly to the Schroedinger picture in which $\Psi_S(t) = \hat{U}(t)\Psi(0)$, one can define an operator $\hat{U}_I(t)$ such that $\Psi_I(t) = \hat{U}_I(t)\Psi(0)$. From eq. (54) we have

$$
\Psi_I(t) = \hat{U}_0^\dagger(t)\hat{U}(t)\Psi(0) \ . \tag{56}
$$

Hence $\hat{U}_I(t) = \hat{U}_0^\dagger(t)\hat{U}(t)$. Substitution of eq. (56) into eq. (55) gives

$$
-\frac{\hbar}{i}\frac{d\hat{U}_I(t)}{dt} = \hat{V}_I(t)\hat{U}_I(t) \ . \tag{57}
$$

The initial condition for the operator $\hat{U}_I(t)$ is $\hat{U}_I(0) = 1$.