QUANTUM PHYSICS III

Solutions to Problem Set 12 3 December 2024

1. General solution of the Dirac equation

1. Substituting the Ansatz
$$\Psi_D = e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-\omega_P t)} u_P \tag{1}$$
into the Dirac equation,
$$-\frac{\hbar}{i}\frac{\partial \Psi_D}{\partial t} = H_D \Psi_D \,, \quad H_D = \sum_{i=1}^{3} \alpha_i p_i + \beta m \tag{2}$$
with
$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \,, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \,, \tag{3}$$
we arrive at
$$(\alpha_i p_i + \beta m) u_P = \omega_P u_P \,. \tag{4}$$

2. Eq. (4) is a homogeneous system of four linear equations which is written in the matrix form as follows,
$$A u_p = 0 \,, \quad A = \begin{pmatrix} m - \omega_P & p_i \sigma_i \\ p_i \sigma_i & -m - \omega_P \end{pmatrix} \,. \tag{5}$$
A nontrivial solution of this system exists if and only if the determinant of A vanishes, that is,
$$\det A = (m^2 + p_1^2 + p_2^2 + p_3^2 - \omega_P^2)^2 = 0 \,, \tag{6}$$
from where it follows that
$$\omega_P = \pm \sqrt{m^2 + p^2} \,. \tag{7}$$
This is nothing but the relation between the momentum and the energy the on-shell particle must obey.

3. In the notation $u_P = (\phi_P \ \chi_P)^T$ the system (5) is written as
$$\begin{aligned} p_i \sigma_i \, \phi_P &= (\omega_P + m) \chi_P \,, \\ p_i \sigma_i \, \chi_P &= (\omega_P - m) \phi_P \,. \end{aligned} \tag{8}$$
From this, the general solution of the Dirac equation is read off straightforwardly. It is
$$u_P = (\phi_P \,, \ (p_i \sigma_i)^{-1} (\omega_P - m) \phi_P)^T \tag{9}$$
with ϕ_P an arbitrary two-dimensional vector.

4. In the non-relativistic limit $\omega_P \approx m$, hence $\chi_P \approx 0$, and the general solution (9) becomes
$$u_P = (\phi_P, \ 0)^T \,. \tag{10}$$

2. Properties of the Dirac matrices

1. The transformation
$$\alpha_i' = U\alpha_i U^{-1}\ , \quad \beta' = U\beta U^{-1}\ , \tag{11}$$
with U a unitary matrix, does not spoil any properties of the Dirac matrices α_i and β. To prove this, note first that their commutation relations remain unchanged, since, for example,
$$\alpha_i'\alpha_j' + \alpha_j'\alpha_i' = U(\alpha_i U^{-1}U\alpha_j + \alpha_j U^{-1}U\alpha_i)U^{-1} = 2UU^{-1}\delta_{ij} = 2\delta_{ij}\,. \tag{12}$$
Next,
$$\alpha_i'^{\dagger} = (U\alpha_i U^{-1})^{\dagger} = (U^{-1})^{\dagger}\alpha_i^{\dagger}U^{\dagger} = U\alpha_i^{\dagger}U^{-1} = \alpha_i'\ , \tag{13}$$
and similarly $\beta'^{\dagger} = \beta'$. Hence, hermiticity is also preserved. Finally,
$$\mathrm{Tr}\ \alpha_i' = \mathrm{Tr}\ (U\alpha_i U^{-1}) = \mathrm{Tr}\ (\alpha_i U^{-1}U) = \mathrm{Tr}\ \alpha_i = 0\ , \tag{14}$$
and similarly $\mathrm{Tr}\,\beta' = 0$.

2. To obtain the Weyl representation, one can choose the transformation matrix as
$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -I & I \\ -I & -I \end{pmatrix}\,. \tag{15}$$
Then,
$$\alpha_i' = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad \beta' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{16}$$

3. Taking the massless limit $m = 0$ in the Weyl representation of the Dirac equation, we have
$$-\frac{\hbar}{i}\frac{\partial\Psi_D}{\partial t} = H_D\Psi_D\,, \quad H_D = \alpha_i' p_i = \begin{pmatrix} -\sigma_i p_i & 0 \\ 0 & \sigma_i p_i \end{pmatrix}. \tag{17}$$
Upon substituting $\Psi_D = (\phi,\ \chi)^T$, the equation above is spitted into two independent equations :
$$\begin{aligned} (i\hbar\partial_t + \sigma_i p_i)\phi &= 0\,, \\ (i\hbar\partial_t - \sigma_i p_i)\chi &= 0\,. \end{aligned} \tag{18}$$
These are the Weyl equations for massless particle. Hence, the advantage of using the Weyl representation of the Dirac matrices is that it allows to disentangle the components of the Dirac spinor Ψ_D in the massless limit.
Multiplying the first equation of the system (18) by $i\hbar\partial_t - \sigma_i p_i$ and the second — by $i\hbar\partial_t + \sigma_i p_i$, we obtain
$$(-\hbar^2\partial_t^2 - p^2)\phi = (-\hbar^2\partial_t^2 - p^2)\chi = 0\ . \tag{19}$$
These are nothing but the Klein-Gordon equation on the Fourier components of Ψ_D. For the particle propagating along the x-line, $p_y = p_z = 0$, and eqs. (18) become
$$\begin{aligned} (i\hbar\partial_t + \sigma_1 p_x)\phi &= 0\,, \\ (i\hbar\partial_t - \sigma_1 p_x)\chi &= 0\,. \end{aligned} \tag{20}$$
The general solution of this system is
$$\phi = c_1 e^{\frac{i}{\hbar}p_x t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-\frac{i}{\hbar}p_x t}\begin{pmatrix}1\\-1\end{pmatrix}\ , \quad \chi = d_1 e^{\frac{i}{\hbar}p_x t}\begin{pmatrix}1\\-1\end{pmatrix} + d_2 e^{-\frac{i}{\hbar}p_x t}\begin{pmatrix}1\\1\end{pmatrix}\ . \tag{21}$$

4. In looking for a new representation of the Dirac matrices, we would like to keep their form as block-diagonals of the Pauli matrices. Then, the requirements of $\beta'^2 = 1$ and $\mathrm{Tr}\,\beta = 0$ constrain the choice of β' to

$$\beta' = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} , \quad \beta' = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} . \tag{22}$$

Let

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \tag{23}$$

be the transformation matrix with u_1, ... u_4 some appropriate 2×2 matrices. Then, it is easy to see that the equation

$$\beta' = U \beta U^{-1} , \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{24}$$

rules out the second possibility in (22) as it requires $u_i = \pm\sigma_2 u_i$ for all i. The first possibility is acceptable and it demands

$$u_3 = \sigma_2 u_1 , \quad u_4 = -\sigma_2 u_2 . \tag{25}$$

Therefore, the matrix U can be rewritten in the form

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} u_1 & u_2 \\ \sigma_2 u_1 & -\sigma_2 u_2 \end{pmatrix} . \tag{26}$$

We put the overall coefficient to ensure that U becomes unitary when u_1 and u_2 are both unitary.
Now we have to choose u_1 and u_2 such that

$$\mathrm{Im}\,(U \alpha_i U^\dagger) = 0 , \quad i = 1, 2, 3 . \tag{27}$$

This is achieved, for example, by taking

$$u_1 = i\sigma_2 , \quad u_2 = I . \tag{28}$$

With this choice, the matrices α'_i become

$$\alpha'_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} , \quad \alpha'_2 = -i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} , \quad \alpha'_3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} . \tag{29}$$

This representation of the Dirac matrices is named after Majorana. It is used in description of a certain type of neutral particles, e.g., Majorana neutrinos.

3. One useful relation

From the commutation relations of the Pauli matrices,

$$\begin{aligned} [\sigma_i,\ \sigma_j] &= 2i\epsilon_{ijk}\sigma_k , \\ \left\{\sigma_i,\ \sigma_j\right\} &= 2\delta_{ij} I , \end{aligned} \tag{30}$$

where I is a unit 2×2 matrix, we have

$$[\sigma_i,\ \sigma_j] + \left\{\sigma_i,\ \sigma_j\right\} = 2(i\epsilon_{ijk}\sigma_k + \delta_{ij}I) = 2\sigma_i\sigma_j\,, \tag{31}$$

hence

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}I\,. \tag{32}$$

Contracting this with the vector $\vec{\pi}$, we obtain

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi}) = \vec{\pi}^2 I + i\vec{\sigma}(\vec{\pi}\times\vec{\pi})\,. \tag{33}$$

To compute the last product, we write the vector $\vec{\pi}$ in components,

$$\begin{aligned}(\vec{\pi}\times\vec{\pi})_i &= \epsilon_{ijk}\left(-i\hbar\partial_j - \frac{e}{c}A_j\right)\left(-i\hbar\partial_k - \frac{e}{c}A_k\right)\\ &= i\frac{\hbar e}{c}\epsilon_{ijk}((\partial_j A_k) + A_j\partial_k)\\ &= i\frac{\hbar e}{c}(\mathrm{rot}\,\vec{A})_i = i\frac{\hbar e}{c}B_i\,.\end{aligned} \tag{34}$$

Substituting this into eq. (33) gives

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi}) = \vec{\pi}^2 I - \frac{\hbar e}{c}\vec{\sigma}\cdot\vec{B}\,. \tag{35}$$

4. On Landau levels

1. The potentials $\vec{A}$ and Φ are related to the magnetic and electric fields as follows,

$$\vec{B} = \mathrm{rot}\,\vec{A}\,,\quad \vec{E} = -\vec{\nabla}\Phi\,. \tag{36}$$

   Taking

$$\vec{A} = -\begin{pmatrix} y\mathcal{B} \\ 0 \\ 0 \end{pmatrix}\,,\quad \Phi = 0\,, \tag{37}$$

   we have in components

$$B_i = \epsilon_{ijk}\partial_j A_k = -\epsilon_{zyx}\mathcal{B}\delta_{zi} = \epsilon_{xyz}\mathcal{B}\delta_{zi} = \mathcal{B}\delta_{zi}\,,\quad E_i = 0\,. \tag{38}$$

   Because of the antisymmetry of ϵ_{ijk}, this result remains unchanged if we shift the vector potential $\vec{A}$ by

$$\vec{A} \to \vec{A} + \vec{\nabla}\alpha(\vec{x})\,, \tag{39}$$

   where α is an arbitrary function of $\vec{x}$. For example, choosing $\alpha = xy\mathcal{B}$, we obtain an equivalent configuration

$$\vec{A}' = \begin{pmatrix} 0 \\ x\mathcal{B} \\ 0 \end{pmatrix}\,,\quad \Phi' = 0\,. \tag{40}$$

   It is clear also that both $\vec{A}$ and Φ can be shifted by an arbitrary constant.

2. We plug the expression for the Dirac spinor,

$$\Psi = e^{-\frac{i}{\hbar}Et}\begin{pmatrix}\phi\\ \chi\end{pmatrix} , \tag{41}$$

into the equation

$$i\frac{\partial\Psi}{\partial t} = (\vec{\alpha}\cdot\vec{\pi} + \beta m + e\Phi)\,\Psi . \tag{42}$$

The result is

$$\begin{aligned}(E - m - e\Phi)\phi &= \vec{\sigma}\cdot\vec{\pi}\,\chi ,\\ (E + m - e\Phi)\chi &= \vec{\sigma}\cdot\vec{\pi}\,\phi .\end{aligned} \tag{43}$$

Expressing χ through ϕ in the second equation of (43) and substituting it to the first equation, we arrive at

$$(E^2 - m^2)\phi = (\vec{\sigma}\cdot\vec{\pi})^2\phi . \tag{44}$$

Let us rewrite the r.h.s. of this equation by the means of eqs. (35) and (37),

$$\begin{aligned}(\vec{\sigma}\cdot\vec{\pi})^2 &= \vec{\pi}^2 - e\vec{\sigma}\cdot\vec{B}\\ &= -\vec{\nabla}^2 + ey^2\mathcal{B}^2 - 2iey\mathcal{B}\partial_x - e\mathcal{B}\sigma_3 .\end{aligned} \tag{45}$$

Hence, eq. (44) becomes

$$(E^2 - m^2)\phi = (-\vec{\nabla}^2 + ey^2\mathcal{B}^2 - 2iey\mathcal{B}\partial_x - e\mathcal{B}\sigma_3)\phi . \tag{46}$$

3. Noticing that the coordinates x and z do not appear in eq. (46) except through the derivatives, one can write the solution as

$$\phi = e^{i(p_x x + p_z z)} f(y) . \tag{47}$$

There will be two independent solutions for $f(y)$ which can be taken, without loss of generality, to be the eigenstates of σ_3 with eigenvalues ± 1. This means that we choose the two independent functions in the form

$$f_+(y) = \begin{pmatrix} F_+(y)\\ 0\end{pmatrix} , \quad f_-(y) = \begin{pmatrix} 0\\ F_-(y)\end{pmatrix} . \tag{48}$$

Since $\sigma_3 f_\pm(y) = \pm f_\pm(y)$, the differential equations satisfied by $F_\pm$ are

$$\frac{d^2F_\pm}{dy^2} + (E^2 - m^2 - p_z^2 \pm e\mathcal{B})F_\pm - (p_x + ey\mathcal{B})^2 F_\pm = 0 . \tag{49}$$

4. The change of variable

$$\xi = \sqrt{e\mathcal{B}}\left(y + \frac{p_x}{e\mathcal{B}}\right) \tag{50}$$

brings eqs. (49) to the form

$$\left(\frac{d^2}{d\xi^2} - \xi^2 + \alpha_\pm\right)F_\pm(\xi) = 0 , \tag{51}$$

where

$$\alpha_{\pm} = \frac{E^2 - m^2 - p_z^2 \pm e\mathcal{B}}{e\mathcal{B}} \ . \tag{52}$$

This is a special form of Hermite's equation, and the solutions exist provided that $\alpha_{\pm} = 2n + 1$ for $n = 0, 1, 2, ...$ This provides the energy eigenvalues

$$E_N^2 = m^2 + p_z^2 + 2Ne\mathcal{B} \ . \tag{53}$$

This is the relativistic form of Landau energy levels. They are two fold degenerate in general : choosing $n = N - 1$ for the "+" sign yields the same energy level as choosing $n = N$ for the "−" sign. Also, because n is non-negative, the ground level $N = 0$ is not degenerate.