Plasma II - Exercises

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Solutions to problem set 9 - May 2, 2025

Multi-choice questions

What process sets the lower limit of the mass of main sequence stars to approximately a tenth of the mass of our sun?

( ) There is no lower limit, but since the luminosity depends strongly on the mass (power of 3-4), light stars radiate too little power to be detected.

(x) Since main sequence stars are gravitationally confined clouds of hydrogen, their temperature increases with their mass and lighter clouds of hydrogen are not sufficiently hot to initiate fusion reactions.

( ) There is a lower mass limit for gas to be gravitationally confined.

( ) Light stars have already burnt all their fuel and extinguished a long time ago.

At least hand-wavingly. A more rigorous argument, as presented in N. Meyer-Vernet, Section 3.1.4) must also take into account the Fermi energy, which limits the contraction of the system.

What happens when energy is added to a gravitational confined system such as a star, i.e. through increased nuclear heating?

( ) The star heats up, potentially increasing nuclear reaction rates and giving rise to a thermal instability.

( ) Nothing, since the temperature depends primarily on the mass of the system.

(x) The star cools down, as an increase of the system's energy increases the radius and hence cools down the star.

This is somewhat counter-intuitive and can be described by a negative heat capacity. It can be easily shown using the virial theorem (also subject of exercise 1).

Exercise 1 - On the role of gravitational energy for the lifetime of our Sun

a) Let us start by demonstrating the *virial theorem*.
At a distance r from the center, a unit volume is subjected to the net outward pressure force $-dP/dr$, and to the inward gravitational force $\rho M_r G/r^2$, where ρ is the density and M_r the mass within the radius r.
Imposing a hydrostatic equilibrium (the balance between the two forces) we obtain,

$$\frac{dP}{dr} = -\frac{\rho M_r G}{r^2} \ . \tag{1}$$

Multiplying both sides by $4\pi r^3$ and integrating in dr over $[0, R]$, we obtain,

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = -\int_0^R \frac{\rho M_r G}{r^2} 4\pi r^3 dr \ . \tag{2}$$

Integrating the left-hand side by parts, we get,

$$-3\int_0^R P(r) 4\pi r^2 dr = -\int_0^R \frac{\rho M_r G}{r} 4\pi r^2 dr \ , \tag{3}$$

where we have considered $P(R) = 0$. We see that on the left-hand side there is three times the average pressure times the volume of the object, while the term on the right-hand side corresponds to the opposite of the gravitational energy,

$$\begin{cases} \int_0^R P(r) 4\pi r^2 dr = \int P dV = \frac{\int P dV}{V} V = \langle P \rangle V \\ -\int_0^R \frac{\rho M_r G}{r} 4\pi r^2 dr = E_g \end{cases} \tag{4}$$

$$\Rightarrow \quad 3\langle P \rangle V = -E_g \ . \tag{5}$$

b) Assuming an ideal gas of N particles with $\gamma = 3$ degrees of freedom inside the Sun, we can write,

$$\langle E_{\rm th} \rangle = \frac{\gamma}{2} N k_B \langle T \rangle = \frac{3}{2} N k_B \langle T \rangle \quad \Rightarrow \quad \langle P \rangle V = N k_B \langle T \rangle = \frac{2}{3} \langle E_{\rm th} \rangle \ . \tag{6}$$

It then follows that

$$-2\langle E_{\rm th} \rangle = E_g \ . \tag{7}$$

So the total amount of energy of the Sun neglecting the contribution coming from nuclear reactions is,

$$E_{\rm tot} = \langle E_{\rm th} \rangle + E_g = \frac{E_g}{2} \ , \tag{8}$$

which is negative, meaning that the object is bound. The total energy is therefore,

$$|E_{\rm tot}| = \frac{E_{\rm g}}{2} = \frac{1}{2}\int_0^R \frac{\rho M_r G}{r} 4\pi r^2 dr = \frac{1}{6}(4\pi\bar{\rho})^2 G \int_0^R r^4 dr = \frac{1}{30}(4\pi\bar{\rho})^2 G R^5 =$$

$$= \frac{3}{10}\frac{GM_R^2}{R} = \frac{3 \times 6.67 \times 10^{-11}\text{N m}^2\text{ kg}^{-2} \times 4 \times 10^{60}\text{kg}^2}{10 \times 7 \times 10^8\text{m}} \simeq 1.1 \times 10^{41}\text{J} \tag{9}$$

where we have used an average value of density $\bar{\rho} = \frac{M_R}{4\pi R^3/3}$.
We can now estimate the lifetime of the Sun τ related to this energy knowing the luminosity,

$$L = -\frac{dE}{dt} \quad \Rightarrow \quad \tau \sim \frac{|E_{\text{tot}}|}{L} = \frac{1.1 \times 10^{41}\text{J}}{4 \times 10^{26}\text{J s}^{-1}} \simeq 10^{14}\text{s} \simeq 10\text{ Myr}\ , \tag{10}$$

which is clearly less than the current age of the Sun. This is the typical life time of a star undergoing Kelvin-Helmholtz contraction, i.e. the phase when the star has run out of fuel for nuclear reactions.

To calculate the energy contribution from nuclear reactions we should first determine the release of energy per proton, which is,

$$\Delta E_n = \Delta m\ c^2 = \frac{(4m_p - m_{He})c^2}{4} =$$

$$= \frac{7 \times 10^{-3} \times 1.67 \times 10^{-27}\text{kg} \times (3 \times 10^9\text{m/s})^2}{4} \simeq 2.6 \times 10^{-11}\text{J}\ . \tag{11}$$

We can now obtain an estimate of the number of protons of the Sun as $M_R/m_p \simeq 10^{57}$ protons. Knowing that only 10% of them are available for nuclear reactions, the total available energy is then:

$$E_n = \Delta E_n N_S/10 = 2.6 \times 10^{-11}\text{J} \times 10^{56} = 2.6 \times 10^{45}\text{J} \tag{12}$$

We see that it is 10^4 times higher than the gravitational energy.

Exercise 2 - Convective speed at the Sun photosphere

Using the mixing-length approach, let us consider a blob which is in pressure equilibrium with the external medium but at a higher temperature, so the relation $\rho T \propto P$ indicates that it has a lower density with respect to its surroundings. This generates the buoyancy force that determines the upward motion of hotter blobs.

Since pressure and temperature decrease as we go upwards, a rising hot blob will encounter a smaller external pressure which makes it expand, so that its temperature decreases. If this temperature variation is less than the environmental temperature decrease, the blob remains hotter than its surroundings and continues to rise.

a) Let's start evaluating the pressure profile characteristic length $H \equiv \left|\frac{P}{dP/dr}\right|$ using the ideal gas equation in the pressure gradient equation for an hydrostatic equilibrium,

$$\frac{dP}{dr} = -\rho g \underbrace{=}_{P=\rho k_B T/m_p} -\frac{m_p P g}{k_B T} = -\frac{1}{H}P \quad \text{with} \quad H = \frac{k_B T}{m_p g}\ . \tag{13}$$

We can now try to assume a constant temperature and gravitational field, and see *a posteriori* if it was a reasonable assumption with the scale length we find. Integrating the previous equation yields,

$$P(r) = \exp\left(- \frac{r}{H} \right) . \tag{14}$$

b) The kinetic energy density reached by the blob when it dissolves is equal to the density of work done by the buoyancy force along ℓ,

$$\frac{1}{2}\rho v_b^2 \sim \Delta\rho_{\mathrm{conv}} g\ell \quad \Rightarrow \quad v_b = v_{\mathrm{conv}} \sim \sqrt{\frac{2\Delta\rho_{\mathrm{conv}} g\ell}{\rho}} , \tag{15}$$

still assuming that g remains constant. We can now use the ideal gas relation $P = \rho k_B T/m_p$ (assuming pure hydrogen plasma) and differentiate it, remembering the hypothesis of pressure balance of the blob with the surrounding environment ($\Delta P = 0$),

$$\Delta\rho_{\mathrm{conv}} T + \rho\Delta T_{\mathrm{conv}} = 0 \quad \Rightarrow \quad \frac{\Delta\rho_{\mathrm{conv}}}{\rho} = -\frac{\Delta T_{\mathrm{conv}}}{T} \quad \Rightarrow \quad v_{\mathrm{conv}} = \sqrt{\frac{2|\Delta T_{\mathrm{conv}}| g\ell}{T}} . \tag{16}$$

As suggested in the problem set, the length of the path travelled by the blob is assumed to be of the order of the characteristic pressure scale length ($\ell \sim H$),

$$v_{\mathrm{conv}} \sim \sqrt{\frac{2|\Delta T_{\mathrm{conv}}| gH}{T}} \sim \sqrt{\frac{2|\Delta T_{\mathrm{conv}}| k_B}{m_p}} . \tag{17}$$

c) To compute ΔT_{conv}, we can consider the transfer of energy per volume with the change of enthalpy, as indicated in the problem set, with the corresponding energy flux given by,

$$F_{\mathrm{conv}} = \left(\frac{1}{m_p}\rho\right) \frac{\gamma}{\gamma - 1} \left(k_B\, \Delta T_{\mathrm{conv}}\right) v_{\mathrm{conv}} = \rho \frac{\gamma}{\gamma - 1} \sqrt{2} \left(\frac{|\Delta T_{\mathrm{conv}}| k_B}{m_p}\right)^{3/2} . \tag{18}$$

This energy flux cannot exceed the radiation flux at the sun's surface that is determined by its luminosity,

$$F_{\mathrm{conv}} \leq F_{\mathrm{lum}} \equiv \frac{L}{4\pi r^2} . \tag{19}$$

Using Eq. 18, we obtain,

$$\Delta T_{\mathrm{conv}} \leq \left[\frac{L(\gamma - 1)}{4\pi r^2 \rho\gamma\sqrt{2}}\right]^{2/3} \frac{m_p}{k_B} . \tag{20}$$

d) Using the provided values we find,

$$H = \frac{k_B T}{m_p g} = \frac{k_B T R_S^2}{m_p M_S G} =$$

$$= \frac{1.38 \times 10^{-23}\mathrm{J/K} \times 10^{6}\mathrm{K} \times (7 \times 10^{8}\mathrm{m})^{2}}{1.67 \times 10^{-27}\mathrm{kg} \times 2 \times 10^{30}\mathrm{kg} \times 6.67 \times 10^{-11}\mathrm{N\ m^{2}\ kg^{-2}}} \simeq 3 \times 10^{7}\ \mathrm{m} \simeq 0.1 R_S\ , \quad (21)$$

where we have used $g = M_S G / R_S^2$. The resulting value of H suggests that the initial assumption of constant T and g along H may not be justified. The corresponding temperature difference is

$$\Delta T_{\mathrm{conv}} \leq \left[\frac{3.84 \times 10^{26}\mathrm{J\ s^{-1}} \times (5/3 - 1)}{4\pi \times (0.7 \times 7 \times 10^{8}\mathrm{m})^{2} \times 200\mathrm{kg\ m^{-3}} \times 5/3 \times \sqrt{2}} \right]^{2/3} \frac{1.67 \times 10^{-27}\mathrm{kg}}{1.38 \times 10^{-23}\mathrm{J\ K^{-1}}} \simeq 0.4\mathrm{K}\ , \quad (22)$$

which is extremely small and the corresponding convection velocity is,

$$v_{\mathrm{conv}} \sim \sqrt{\frac{2 \times 0.4\mathrm{K} \times 1.38 \times 10^{-23}\mathrm{J\ K^{-1}}}{1.67 \times 10^{-27}\mathrm{kg}}} \simeq 81\ \mathrm{m/s}\ . \quad (23)$$