Plasma II - Exercises

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Solutions to problem set 2 - February 28, 2025

Exercise 1 - The Bennett Z-pinch

a) The general form of the current density and magnetic field of a Z-pinch is,

$$\begin{cases} \vec{B} = [0, B_\theta(r), 0] \\ \vec{j} = [0, 0, j_z(r)] \end{cases} .$$

Using Ampère's law we obtain the relation between the current density and the magnetic field,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \underbrace{\Rightarrow}_{\partial_z = \partial_\theta = 0} \mu_0 j_z(r) = \frac{1}{r}\frac{\partial}{\partial r}[r B_\theta(r)] = \frac{1}{r}\left[B_\theta(r) + r\frac{\partial B_\theta(r)}{\partial r}\right] , \qquad (1)$$

which can be subsequently used to replace the current density in the ideal MHD force balance equation,

$$\vec{j} \times \vec{B} = \vec{\nabla} p \quad \Rightarrow \quad \frac{\partial p(r)}{\partial r} = -B_\theta(r) j_z(r) = -\frac{B_\theta(r)}{\mu_0}\left[\frac{B_\theta(r)}{r} + \frac{\partial B_\theta(r)}{\partial r}\right] \quad \Rightarrow$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left[p(r) + \frac{B_\theta^2(r)}{2\mu_0}\right] + \frac{B_\theta^2(r)}{\mu_0 r} = 0 . \qquad (2)$$

b) The radial current density profile is assumed to be,

$$j_z(r) = \begin{cases} \frac{2I}{\pi}\frac{a^2}{(r^2+a^2)^2} & \text{for } r \leq a \\ 0 & \text{for } r > a \end{cases} .$$

The corresponding **poloidal magnetic field** can be caluclated by integrating Eq. (1). For $r \leq a$,

$$B_\theta(r) = \frac{\mu_0}{r}\int_0^r r' j_z(r') dr' = \frac{2\mu_0 I a^2}{r\pi}\int_0^r \frac{r'}{(r'^2 + a^2)^2} dr' .$$

This integral can be easily solved performing a change of variables $r'^2 + a^2 = b$,

$$\int_0^r \frac{r'}{(r'^2+a^2)^2} dr' \underbrace{=}_{r'^2+a^2=b} \int_{a^2}^{r^2+a^2} \frac{1}{2b^2} db = \left(-\frac{1}{2b}\right)\Bigg|_{a^2}^{r^2+a^2} = \frac{r^2}{2a^2(r^2+a^2)} .$$

For $r > q$ the integrand becomes zero and the integral remains constant.

$$\begin{cases} r \le a: & \int_0^r \frac{r'}{(r'^2+a^2)^2} dr' = \frac{r^2}{2a^2(r^2+a^2)} \\ r > a: & \int_0^a \frac{r'}{(r'^2+a^2)^2} dr' = \frac{1}{4a^2} \end{cases}$$

It then follows,

$$B_\theta(r) = \begin{cases} \frac{\mu_0 I}{\pi} \frac{r}{r^2+a^2} & \text{for } r \le a \\ \frac{\mu_0 I}{2\pi r} & \text{for } r > a \end{cases} . \quad (3)$$

Note that the magnetic field outside the plasma column can be easily obtained without knowledge of the radial current profile by integrating Ampère's law over a cross section of the plasma column and using Stokes theorem,

$$\nabla \times \vec{B} = \mu_0 \vec{j} \quad \Rightarrow \quad \int_C \vec{B} \cdot d\vec{\ell} = 2\pi r B_\theta(r) = \mu_0 \int_S \vec{j} \cdot d\vec{S} = \mu_0 I$$

$$\Rightarrow \quad B_\theta(r) = \frac{\mu_0 I}{2\pi r}$$

where C is the contour of integration of the surface S.

We can now compute the **pressure profile** from Eq. (2),

$$p(r) - p(r=0) = -\frac{B_\theta^2(r)}{2\mu_0} - \frac{1}{\mu_0} \int_0^r \frac{B_\theta^2(r')}{r'} dr' .$$

Substituting Eq. (3) for $r \le a$ (the pressure is of course inside the plasma column only!), the integral becomes,

$$\int_0^r \frac{B_\theta^2(r')}{r'} dr' = \int_0^r \left(\frac{\mu_0 I}{\pi}\right)^2 \frac{r'}{(r'^2+a^2)^2} dr' = \left(\frac{\mu_0 I}{\pi}\right)^2 \frac{r^2}{2a^2(r^2+a^2)}$$

using the same change of variables of the previous calculation. The plasma pressure radial profile is then given by,

$$p(r) = p_0 - \frac{\mu_0 I^2}{2\pi^2} \left(\frac{r^2}{r^2+a^2}\right)\left(\frac{1}{a^2} + \frac{1}{r^2+a^2}\right) = p_0 - \frac{\mu_0 I^2}{2\pi^2} \frac{r^2(r^2+2a^2)}{a^2(r^2+a^2)^2} .$$

If we now assume a pressure equal to zero at $r = a$, we obtain,

$$p(r=a) = p_0 - \frac{\mu_0 I^2}{2\pi^2} \frac{3}{4a^2} = 0 \quad \Rightarrow \quad p_0 = \frac{3\mu_0 I^2}{8\pi^2 a^2} .$$

The complete expression for the pressure radial profile is,

$$p(r) = \frac{3\mu_0 I^2}{8\pi^2 a^2} \left[1 - \frac{4r^2(r^2+2a^2)}{3(r^2+a^2)^2}\right] . \quad (4)$$

We can now perform a numerical application using $I = 1$ kA and $a = 1$m to compute the values of j_0, B_0 and p_0,

$$j_0 = \frac{2I}{\pi a^2} = \frac{2 \times 10^3 \text{A}}{\pi \times 1\text{m}^2} \simeq 637\,\text{A/m}^2 , \quad B_0 = 0 , \quad p_0 = \frac{3\mu_0 I^2}{8\pi^2 a^2} \simeq 4.8 \times 10^{-2}\,\text{A}^2/\text{m}^2 .$$

The normalised profiles are shown in Fig. 1.

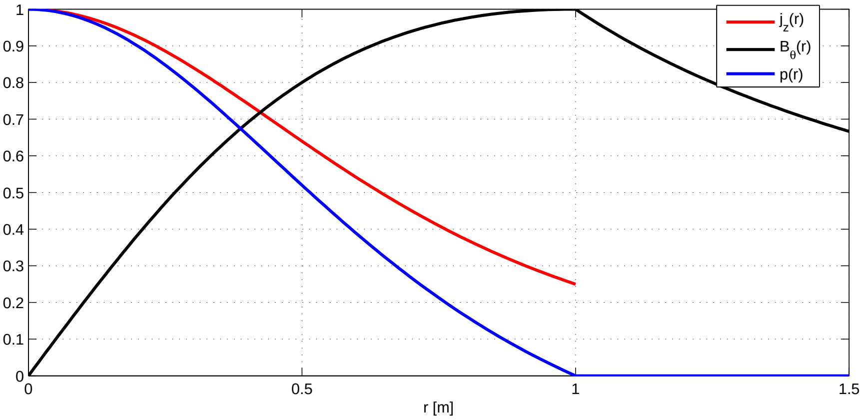

Figure 1: Assumed axial current density profile (red) and resulting poloidal magnetic field (black) and pressure (blue) profiles of a Z-pinch equilibrium.

c) The total **thermal energy** is[1],

$$W_{\text{th}} = \int \frac{p}{\gamma - 1}\, dV = \frac{3}{2}\int_0^{2\pi} d\theta \int_0^L dz \int_0^a dr\, p(r)\, r \ .$$

Using the radial pressure profile, Eq. 4, determined in b), the expression for the thermal energy becomes,

$$W_{\text{th}} = 3\pi L \frac{3\mu_0 I^2}{8\pi^2 a^2} \int_0^a dr \left[1 - \frac{4r^2(r^2 + 2a^2)}{3(r^2 + a^2)^2}\right] r \ . \tag{5}$$

The integral of the first term in the parenthesis gives simply $a^2/2$. The second term requires more calculations,

$$- \int_0^a dr \frac{4r^3(r^2 + 2a^2)}{3(r^2 + a^2)^2} \underbrace{=}_{x=r/a} - \frac{4a^2}{3} \int_0^1 x^3 \frac{(x^2 + 2)}{(1 + x^2)^2} dx =$$

$$= -\frac{4a^2}{3}\left[\int_0^1 \frac{x^5}{(1 + x^2)^2} dx \ + \int_0^1 \frac{2x^3}{(1 + x^2)^2} dx\right] . \tag{6}$$

Both integrals can be solve by integration by parts, and we can actually obtain a generic formula,

$$I_{m,n} = \int \frac{x^m}{(1 + x^2)^n} dx = \int f(x) g'(x) = f(x) g(x) - \int f'(x) g(x)$$

with

$$f(x) = x^{m-1} \qquad g'(x) = \frac{x}{(1 + x^2)^n} \qquad \Rightarrow \qquad g(x) = \frac{1}{2(n-1)(1 + x^2)^{n-1}} \ .$$

[1]The calculation of the thermal energy gets an extra factor $1/(\gamma - 1) = 3/2$ from incorporating the adiabatic condition and then computing the energy equation from the other equations of MHD. See for example the textbook *Fundamentals of plasma physics* of P.M. Bellan (pp. 307-310)

It follows,

$$I_{m,n} = -\frac{x^{m-1}}{2(n-1)(1+x^2)^{n-1}} + \frac{m-1}{2(n-1)} \int dx \frac{x^{m-2}}{(1+x^2)^{n-1}} \ .$$

We can write the expression 6 as,

$$-\frac{4a^2}{3}(I_{5,2} + 2I_{3,2}) \ . \tag{7}$$

The first integral is,

$$I_{5,2} = \int_0^1 \frac{x^5}{(1+x^2)^2} dx = -\frac{x^4}{2(1+x^2)}\bigg|_0^1 + 2\int_0^1 \frac{x^3}{1+x^2} dx =$$

$$= -\frac{1}{4} + x^2 \ln(1+x^2)\bigg|_0^1 - \int_0^1 2x \ln(1+x^2) dx \underbrace{=}_{1+x^2=y} -\frac{1}{4} + \ln 2 - \int_1^2 \ln y dy =$$

$$= -\frac{1}{4} + \ln 2 - \left(y \ln y - y\right)\bigg|_1^2 = \frac{3}{4} - \ln 2 \ .$$

The second integral is,

$$I_{3,2} = \int_0^1 \frac{x^3}{(1+x^2)^2} dx = \left[-\frac{x^2}{2(1+x^2)}\right]\bigg|_0^1 + \int_0^1 \frac{x}{1+x^2} dx =$$

$$= -\frac{1}{4} + \frac{1}{2}\ln(1+x^2)\bigg|_0^1 = -\frac{1}{4} + \frac{1}{2}\ln 2 \ .$$

We can therefore rewrite expression 7 as,

$$-\frac{4a^2}{3}(I_{5,2} + 2I_{3,2}) = -\frac{4a^2}{3}\left(\frac{3}{4} - \ln 2 - \frac{1}{2} + \ln 2\right) = -\frac{a^2}{3} \ .$$

It follows,

$$W_{\text{th}} = \frac{9\mu_0 L I^2}{8\pi a^2}\left(\frac{a^2}{2} - \frac{a^2}{3}\right) = \frac{3\,\mu_0\, L I^2}{16\pi} \ . \tag{8}$$

The **magnetic energy** can be computed as,

$$W_{\text{mag}} = \int \frac{B_\theta^2(r)}{2\mu_0} dV = \frac{\mu_0 L I^2}{\pi} \int_0^a dr \frac{r^3}{(r^2+a^2)^2} = \frac{\mu_0 L I^2}{\pi} \int_0^1 \frac{x^3}{(1+x^2)^2} dx =$$

$$= \frac{\mu_0 L I^2}{4\pi}(2\ln 2 - 1) \ .$$

In this example the ratio of magnetic and thermal energies is,

$$\frac{W_{\text{mag}}}{W_{\text{th}}} = \frac{4}{3}(2\ln 2 - 1) = 0.51 \sim O(1) \ .$$

Exercise 2 - The screw-pinch

a) We can proceed in the very same way of the previous exercise. The general form of the current density and magnetic field of a screw-pinch is,

$$\begin{cases} \vec{B} = [0, B_\theta(r), B_z(r)] \\ \vec{j} = [0, j_\theta(r), j_z(r)] \end{cases} .$$

Using Ampère's law we obtain relations between the components of the current and magnetic field,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} = \mu_0[0, j_\theta(r), j_z(r)] = \left\{0, -\frac{\partial B_z(r)}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}[rB_\theta(r)]\right\} . \tag{9}$$

Use these relations to eliminate the current density in the ideal MHD force balance equation,

$$\vec{j} \times \vec{B} = \vec{\nabla} p \quad \Rightarrow \quad \frac{\partial p(r)}{\partial r} = B_z(r) j_\theta(r) - B_\theta(r) j_z(r) = \tag{10}$$

$$\frac{1}{\mu_0}\left\{ - B_z(r)\frac{\partial B_z(r)}{\partial r} - B_\theta(r)\frac{1}{r}\frac{\partial}{\partial r}[rB_\theta(r)]\right\} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left[p(r) + \frac{B_z^2(r) + B_\theta^2(r)}{2\mu_0}\right] + \frac{B_\theta^2(r)}{\mu_0 r} = 0 .$$

b) The externally applied axial field can be modified by a poloidal plasma current, j_θ. We inspect the sign of this current by solving Eq. 10 for j_θ,

$$j_\theta(r) = \frac{1}{B_z(r)}\left[\frac{\partial p(r)}{\partial r} + j_z(r) B_\theta(r)\right] . \tag{11}$$

Since the pressure profile generally peaks in the plasma centre, the pressure gradient is negative, $\partial p(r)/\partial r < 0$. As the poloidal field is created by the axial current, $j_z B_\theta$ is always positive, $j_z B_\theta > 0$. The sign of the sum in the parenthesis on the R.H.S. of Eq. 11 depends on the magnitude of the pressure gradient. Considering that $B_z(r) > 0$,

$$j_\theta = \begin{cases} > 0 \quad \text{for} \quad \left|\frac{\partial p(r)}{\partial r}\right| < j_z(r) B_\theta(r) \quad \text{, i.e. low plasma pressure} \\ < 0 \quad \text{for} \quad \left|\frac{\partial p(r)}{\partial r}\right| > j_z(r) B_\theta(r) \quad \text{, i.e. high plasma pressure} \end{cases}$$

At low plasma pressure, the poloidal current $j_\theta(r)$ is positive throughout the radial profile and reinforces the toroidal magnetic field $B_z(r)$, thus producing a paramagnetic effect. Without any plasma pressure and, hence, pressure gradient, $j_\theta/j_z = B_\theta/B_z$, which means that current and magnetic field have the same *pitch angle* and are, therefore, aligned and *force-free*. At sufficiently high plasma pressure, the poloidal current changes sign and starts to reduce the toroidal magnetic field, thus producing a diamagnetic effect. Note that the effect on the axial field remains the same when changing the direction of the axial field as j_θ changes its sign with B_z.

c) The generic expression of the rotational transform $\iota(r)$ and safety factor profile $q(r)$ can be obtained from the continuous form of their definitions,

$$\iota = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d\theta_k = \int_0^{2\pi} d\theta = \int_0^L \frac{d\theta}{dz} dz \quad \text{and} \quad q(r) = \frac{2\pi}{\iota(r)} \,. \tag{12}$$

The ratio of poloidal and axial advances along a magnetic field line depends on the corresponding components of the magnetic field,

$$\frac{d\theta}{dz} = \frac{B_\theta(r)}{rB_z(r)} \,.$$

The integration in Eq. 12 is trivial, yielding,

$$\iota(r) = \int_0^L \frac{d\theta(r)}{dz} dz = L\frac{B_\theta(r)}{rB_z(r)} \quad \text{and} \quad q(r) = \frac{2\pi}{\iota(r)} = \frac{2\pi r B_z(r)}{LB_\theta(r)} \,.$$