Plasma II - Exercises

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Solutions to problem set 1 - 21 February 2025

Exercise 1 - Ideal Ignition

a) *Ideal ignition* occurs when the α particle heating, P_α, compensates for bremsstrahlungs losses, P_b. Neglecting He ash in the plasma, Z_eff in the expression of P_b is set to 1,

$$\frac{P_\alpha}{P_\mathrm{b}} = \frac{\frac{1}{4}n^2\langle\sigma v\rangle_{DT}\Delta E_\alpha}{An^2T^{1/2}} = 1 \qquad \begin{cases} \Delta E_\alpha = 3.5\ \mathrm{MeV} \\ A = 5\times10^{-37}\,\frac{\mathrm{Wm^3}}{\sqrt{\mathrm{keV}}} \end{cases}$$

Since P_α and P_b have the same density dependence, the condition depends only on the plasma temperature and yields an *ideal ignition temperature*, T_ideal. Replacing $\langle\sigma v\rangle_\mathrm{DT}$ with the quadratic expansion given in the exercise yields,

$$\frac{\frac{1}{4}\,1.1\times10^{-24}\frac{\mathrm{m^3}}{\mathrm{s\,keV^2}}T_\mathrm{ideal}^2\ 1.6\times10^{-19}\,\frac{\mathrm{J}}{\mathrm{eV}}\ 3.5\times10^6\mathrm{eV}}{5\times10^{-37}\frac{\mathrm{Wm^3}}{\sqrt{\mathrm{keV}}}T_\mathrm{ideal}^{1/2}} = 1\ . \tag{1}$$

Solving Eq. 1 for the temperature results in,

$$T_\mathrm{ideal} = 2.2\,\mathrm{keV}\ .$$

b) The value for T_ideal obtained in a) is clearly outside of the applicable range of the quadratic expansion of $\langle\sigma v\rangle_\mathrm{DT}$ and further terms have to be taken into account. The resulting expression is easily solved graphically, which yields T_ideal of approximately 4.4 keV (see Fig. 1).

c) To study the stability of the system, the plasma energy dynamics can be considered,

$$\frac{dE}{dt} = P_\alpha - P_\mathrm{b} \quad \Rightarrow \quad \frac{d(3nT)}{dt} = \frac{1}{4}n^2\langle\sigma v\rangle_\mathrm{DT}\Delta E_\alpha - An^2T^{1/2}\ . \tag{2}$$

The stability of the system is determined by its response to small perturbations. Since the equilibrium does not depend on n, we only consider a perturbation δT to the equilibrium temperature T_ideal. A linear expansion of the temperature dependent terms in Eq. 2 at $T = T_\mathrm{ideal}$ yields,

$$3n\frac{d(\delta T)}{dt} = \frac{n^2}{4}\left.\frac{d\langle\sigma v\rangle_\mathrm{DT}}{dT}\right|_{T=T_\mathrm{ideal}}\Delta E_\alpha\delta T - \frac{An^2}{2}\frac{1}{T_\mathrm{ideal}^{1/2}}\delta T\ .$$

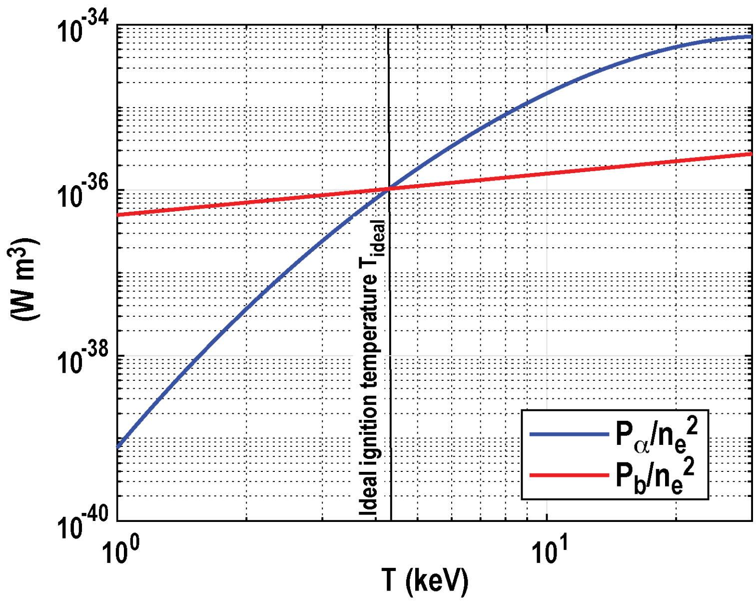

Figure 1: Temperature dependence of the alpha heating power P_α and bremsstrahlung losses P_b (assuming Z_eff=0) yielding the ideal ignition temperature T_ideal, where $P_\alpha = P_\mathrm{b}$.

The analytical expression for $\langle\sigma v\rangle_\mathrm{DT}$ given in the problemset is now used to evaluate its derivative,

$$3n\frac{d(\delta T)}{dt} = \frac{n^2}{4}\,\langle\sigma v\rangle_\mathrm{DT}|_{T=T_\mathrm{ideal}}\left(-\frac{\alpha\, a_{-1}}{T_\mathrm{ideal}^{\alpha+1}} + a_1\right)\Delta E_\alpha \delta T - \frac{An^2}{2}\frac{1}{T_\mathrm{ideal}^{1/2}}\delta T\ .$$

The expression can be simplified using the equilibrium condition $1/4\ \langle\sigma v\rangle_\mathrm{DT}|_{T_\mathrm{ideal}}\,\Delta E_\alpha = AT_\mathrm{ideal}^{1/2}$,

$$\begin{aligned}3\frac{d(\delta T)}{dt} &= nA\left[T_\mathrm{ideal}^{1/2}\left(-\frac{\alpha\, a_{-1}}{T_\mathrm{ideal}^{\alpha+1}} + a_1\right) - \frac{1}{2T_\mathrm{ideal}^{1/2}}\right]\delta T \\ &= \frac{nA}{2T_\mathrm{ideal}^{1/2}}\left[2T_\mathrm{ideal}\left(-\frac{\alpha\, a_{-1}}{T_\mathrm{ideal}^{\alpha+1}} + a_1\right) - 1\right]\delta T\quad . \end{aligned} \tag{3}$$

To verify if the *ideal ignition* temperature is a stable or unstable condition, we have to check if the term in the square brackets on the right-hand side term of Eq. 3 is positive (exponentially growing) or negative (stable). This corresponds to study the following relation:

$$2T_\mathrm{ideal}\left(-\frac{\alpha\, a_{-1}}{T_\mathrm{ideal}^{\alpha+1}} + a_1\right) \qquad \begin{cases} > 1 & \text{unstable} \\ < 1 & \text{stable} \end{cases} \tag{4}$$

Substituting the coefficients into Eq. (4) yields,

$$2T_\mathrm{ideal}\left(-\frac{\alpha\, a_{-1}}{T_\mathrm{ideal}^{\alpha+1}} + a_1\right) \simeq 7 > 1 \quad \Rightarrow \quad \textbf{unstable}\ .$$

Note that an even more complete parametric description of $\langle\sigma v\rangle_{\mathrm{DT}}$ is given in [Freidberg, Plasma Physics & Fusion Energy, page 58],

$$\langle\sigma v\rangle_{DT} = 10^{-6}\,\frac{\mathrm{m}^3}{\mathrm{s}}\ \exp\left(\frac{a_{-1}}{T^\alpha} + a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_4 T^4\right) \tag{5}$$

$$\begin{cases} \alpha = 0.2935 \\ a_{-1} = -21.38 \\ a_0 = -25.20 \\ a_1 = -7.101\times10^{-2} \\ a_2 = 1.928\times10^{-4} \\ a_3 = 4.925\times10^{-6} \\ a_4 = -3.984\times10^{-8} \end{cases}$$

where T is expressed in keV. However, since we want to study the stability around the ideal ignition temperature, we can neglect the higher order terms. Figure 2 shows that for $T \approx 4.4keV$ the omission of the 4^{th}, 3^{rd} and 2^{nd} order terms introduces only a small error in $\langle\sigma v\rangle_{\mathrm{DT}}$.

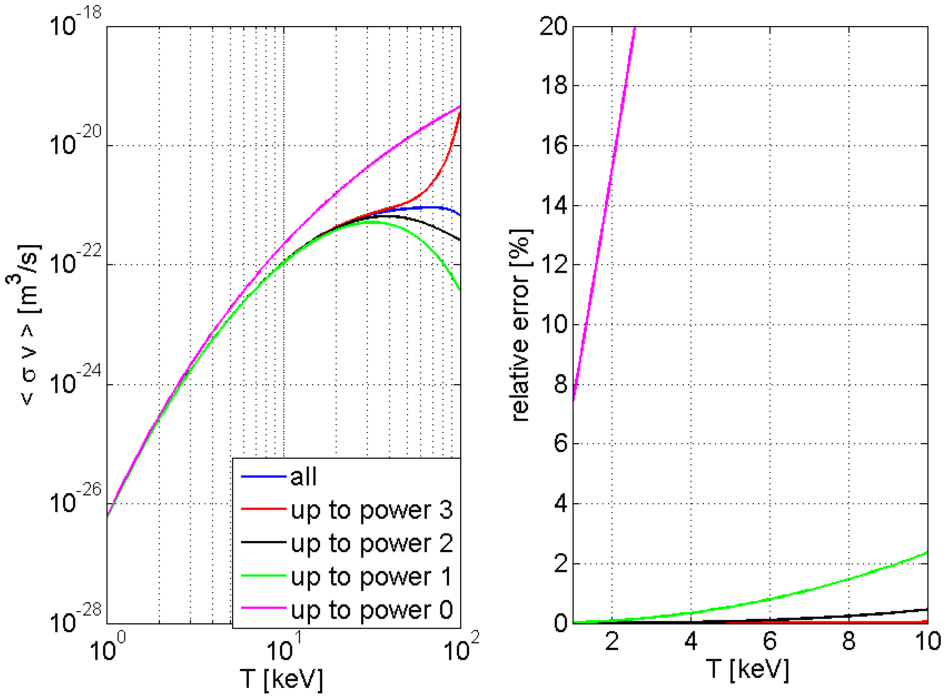

Figure 2: Plot of the analytical expression of $\langle\sigma v\rangle_{\mathrm{DT}}$ previously provided, for several degrees of approximation, with the corresponding relative error with respect to the complete expression.