Plasma II - Exercises

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Solutions to problem set 11 - May 16, 2025

Multiple-choice question

Reconnection converts

( ) kinetic energy into magnetic energy.

( ) magnetic energy into kinetic energy.

(x) magnetic energy into kinetic energy and heat.

Design of an experiment to study the kink instability

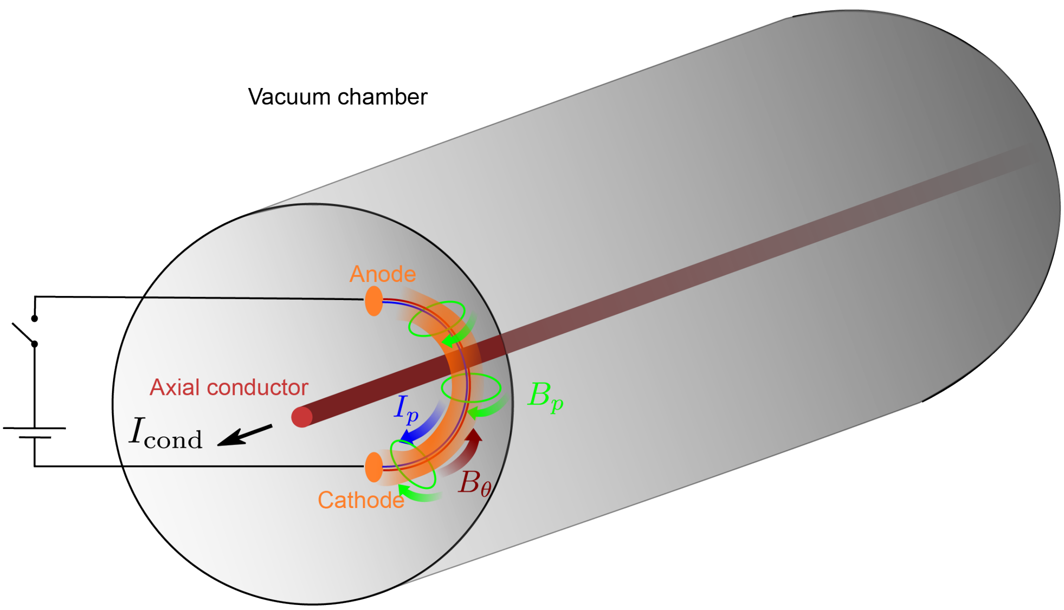

Figure 1: Sketch of the experimental setup.

a) Considering the hardware available, we can place the two electrodes as shown in Figure 1 (in orange). In that way, the magnetic field lines generated by the central conductor (in brown) are perpendicular to the electrode surfaces and confine the plasma produced between the two electrodes. Once the plasma breakdown is obtained, the flux rope (in orange) is poloidally contained (along $\hat{\theta}$) between the anode and the cathode, with the second magnetic field (B_p in green) generated by the plasma current (I_p in blue). A semi-circular flux rope can thus be generated.

b) From the Paschen curve for H_2, we can calculate the pd product corresponding to the minimum voltage to obtain the gas breakdown,

$$pd|_{\min} = 0.15\,\text{mbar} \times 0.1\,\text{m} = 1.5\,\text{Pa}\,\text{m} \;. \tag{1}$$

To compute the distance between the electrodes (along the magnetic field lines), we need to calculate the gas pressure of our experiment, assuming an ideal gas at room temperature (T=300 K),

$$pV = n_m RT \quad \Rightarrow \quad p = \frac{nRT}{\pi r^2 L} = \frac{10^{-3}\,\text{mol} \times 8.31\text{J}/(\text{mol}\,\text{K}) \times 300\,K}{3.14 \times (0.5\,\text{m})^2 \times 2\,\text{m}} \simeq 1.53\,\text{Pa} \;. \tag{2}$$

The corresponding distance between the electrodes is,

$$d = \frac{pd|_{\min}}{p} = \frac{1.5\,\text{Pa}\,\text{m}}{1.53\,\text{Pa}} \simeq 1\,\text{m} \;. \tag{3}$$

c) Let us take the expression for a plasma column safety factor,

$$q(r) = \frac{2\pi r}{\ell}\frac{B_\theta}{B_p} \;, \tag{4}$$

where ℓ is the length of the plasma column and we have kept the same coordinate system of our problem. Since the threshold is given by the safety factor at the edge, we can consider the previous expression at $r = a$,

$$q(a) = q_a = \frac{2\pi a}{\ell}\frac{B_\theta}{B_p(a)} = \frac{(2\pi)^2 a^2}{\ell}\frac{B_\theta}{I_p \mu_0} \;, \tag{5}$$

where we have used Ampere's law to determine the magnetic field corresponding to the plasma current inside the flux rope. Using the threshold condition, we obtain,

$$q_a = 1 = \frac{(2\pi)^2 a^2}{\ell}\frac{B_\theta}{I_p \mu_0} = \frac{I_{K-S}}{I_p} \qquad \text{with} \qquad I_{K-S} = \frac{(2\pi)^2 a^2}{\ell}\frac{B_\theta}{\mu_0} \;. \tag{6}$$

d) To compute the current we study the constraints we have on the system. Let's start from the stability/instability kink condition.
To find the condition for the flux rope to become unstable against the kink instability, we use the *Kruskal-Shafranov* condition $I_p \geq I_{\text{K-S}}$, where I_p is the plasma current

inside the flux rope. As indicated in the problem set, the plasma current can be assumed to be equal to the ion saturation current I_s,

$$I_s = \frac{n_e e c_s A_{\text{eff}}}{2} \geq I_{\text{K-S}} = \frac{(2\pi)^2 a^2 B_\theta}{\mu_0 \ell} \ , \tag{7}$$

where $c_s = \sqrt{k_B T_e / m_i}$ is the ion sound speed, A_{eff} is the effective collecting area of the electrode and B_θ is the poloidal magnetic field along the connection length between the two electrodes. This can be written as,

$$B_\theta = \frac{\mu_0 I_{\text{cond}}}{2\pi r_e} = \frac{\mu_0 I_{\text{cond}}}{2\ell} \tag{8}$$

knowing that the two electrode surfaces are poloidally separated by 180 degrees (they lay on the same plane).
Since it is reasonable to consider the whole electrode surface as collecting area, the *Kruskal-Shafranov* condition, Eq. (7), becomes,

$$\frac{n_e e c_s \pi a^2}{2} \geq \frac{(2\pi)^2 a^2}{\mu_0 \ell} \frac{\mu_0 I_{\text{cond}}}{2\ell} \ . \tag{9}$$

We can obtain the electron density from the number of moles we have, considering the information on the degree of ionization of the gas,

$$n_e = 0.5 n = 0.5 \, \frac{n_m \, N_A}{V} \approx 0.5 \, \frac{10^{-3}\text{mol} \times 6 \times 10^{23} \, \text{part/mol}}{1.5 \, \text{m}^3} \simeq 2 \times 10^{20} \, \text{part/m}^3 \ . \tag{10}$$

The ion sound speed is,

$$c_s = \sqrt{\frac{k_B T_e}{m_i}} = \sqrt{\frac{1.69 \times 10^{-18}\text{J}}{1.67 \times 10^{-27}\text{kg}}} \simeq 3 \times 10^4 \text{m/s} \ . \tag{11}$$

The *Kruskal-Shafranov* condition can be reduced to,

$$I_{\text{cond}} \leq \frac{n_e e \ell^2 c_s}{4\pi} =$$

$$= \frac{2 \times 10^{20} \, \text{par/m}^3 \times 1.69 \times 10^{-19}\text{C} \times (1\text{m})^2 \times 3 \times 10^4 \text{m/s}}{4\pi} \simeq 85 \, \text{kA} \ . \tag{12}$$

We now have to verify for which values of magnetic field the plasma is magnetised.
If we consider the condition on the temporal scale lengths, we have,

$$T = \frac{\ell}{c_s} > \frac{1}{\omega_{ce}} = \frac{m_e}{e B_\theta} = \frac{m_e 2\ell}{e \mu_0 I_{\text{cond}}} \quad \Rightarrow$$

$$\Rightarrow \quad I_{\text{cond}} > \frac{2 m_e c_s}{e \mu_0} = \frac{2 \times 9 \times 10^{-31}\text{kg} \times 3 \times 10^4 \text{m/s}}{1.69 \times 10^{-19}\text{C} \times 4\pi \times 10^{-7}\text{Hm}^{-1}} \simeq 0.25\text{A} \ . \tag{13}$$

For the spatial scale length we have,

$$L_\perp = a > \rho_{ce} = \frac{v_{\rm th}}{\omega_{ce}} = \sqrt{\frac{k_B T_e}{m_e}} \frac{m_e 2\ell}{e\mu_0 I_{\rm cond}} \quad \Rightarrow$$

$$\Rightarrow \quad I_{\rm cond} > \frac{2\ell\sqrt{m_e k_B T_e}}{e\mu_0 a} \simeq \frac{2{\rm m} \times \sqrt{9\times 10^{-31}{\rm kg} \times 1.69 \times 10^{-18}{\rm J}}}{1.69 \times^{-19} {\rm C} \times 4\pi \times 10^{-7}{\rm Hm}^{-1} \times 0.05{\rm m}} \simeq 200{\rm A}\ . \quad (14)$$

It's clear as the condition on the spatial scale length is the most constraining one, so that the resulting range of the axial conductor current is,

$$I_{\rm cond} \in [0.2 \div 85]\,{\rm kA}\ . \quad (15)$$

e) Let us evaluate the temperature increase associated with the Ohmic dissipation inside the central axial conductor. The temperature variation ΔT of the axial conductor for a current I driven for a time Δt, is,

$$\Delta T = \frac{Q}{m_{\rm cond}\, c_{\text{s-copper}}} = \frac{P\Delta t}{V_{\rm cond}\, d_{\rm copper}\, c_{\text{s-copper}}} = \frac{RI^2\Delta t}{\pi r_{\rm cond}^2 L d_{\rm copper}\, c_{\text{s-copper}}} =$$

$$= \frac{\rho_{\rm copper} L I^2 \Delta t}{(\pi r_{\rm cond}^2)^2 L d_{\rm copper}\, c_{\text{s-copper}}} \quad \Rightarrow \quad \Delta T = \frac{\rho_{\rm copper}}{\pi^2 d_{\rm copper}\, c_{\text{s-copper}}} \frac{I^2}{r_{\rm cond}^4} \Delta t\ . \quad (16)$$

Assuming a current of 2 kA for one second, we obtain,

$$\Delta T = \frac{1.5\times 10^{-8}\,\Omega\,{\rm m}}{\pi^2 \times 8.96 \times 10^3\,{\rm kg\,m^{-3}} \times 385\,{\rm J\,kg^{-1}\,K^{-1}}} \frac{(2\times 10^3\,{\rm A})^2}{(0.01{\rm m})^4} \times 1{\rm s} \simeq 0.2\,{\rm K}\ . \quad (17)$$

This results in a negligible temperature increase, since the time scales of the experiments are much smaller than 1 sec. It should however be stressed that continuous (CW) experiments do not involve complex power supply controllers and are, therefore, simpler to implement. On the negative side, CW experiments usually need water cooling of the active parts.
We should underline that the dependence of the temperature variation on the inverse of the forth power of the cross section radius of the conductor. A conductor with 5 mm cross section would have given a heating of 3 K per second (still with 2 kA of current)!

f) This question clearly requires the estimate of the characteristic growth-rate of the unstable kink, that we have seen equal to,

$$\gamma = \frac{v_A \pi}{\ell}\sqrt{\left(\frac{I_p}{I_{\rm K-S}}\right)^2 - 1} = \frac{\pi I_{\rm cond}}{2\ell^2}\sqrt{\frac{\mu_0}{m_i n_i}}\sqrt{\left(\frac{I_p}{I_{\rm K-S}}\right)^2 - 1}\ , \quad (18)$$

where $v_A = B_\theta/\sqrt{\mu_0 m_i n_i}$ is the Alfvèn velocity. Using the provided value of conductor current we end up with,

$$\gamma = \frac{\sqrt{3}\times\pi\times 1.2\times 10^5{\rm A}}{4\times(1m)^2} \times \sqrt{\frac{10^{-7}{\rm H/m}}{1.6\times 10^{-27}{\rm kg}\times 3\times 10^{20}\,{\rm par/m^3}}} \simeq 1.6\times 10^5{\rm s}^{-1}\ . \quad (19)$$

This corresponds to a characteristic time scale of,

$$\tau \sim \frac{1}{\gamma} \simeq 6\mu\text{s} . \tag{20}$$

This suggests that a fast camera should image the kinked flux rope with a time resolution of at least $6/10\,\mu$s (assuming that you are happy with ten frames during the expansion phase).