Plasma II - Exercises

Dr. H. Reimerdes, Dr. E. Tonello - EPFL/SPC

Solutions to problem set 10 - May 9, 2025

Multiple-choice question

Why are sunspots cooler than their surroundings?

(x) The large magnetic field of sun spots suppresses convection and, thereby, the primary heat transport mechanisms from deeper regions towards the surface.

(x) The typical flaring of flux tubes caused by radial gradients in the magnetic field causes transported heat to spread over a greater surface area.

( ) The lower temperature is only an artefact of high magnetic field affecting our temperature measurements.

Why does a dynamic dynamo require a sufficiently high magnetic Reynolds number R_m?

( ) Because MHD theory is not valid at low magnetic Reynolds number.

(x) A low magnetic Reynolds number indicates high resistive dissipation, which primarily leads to a conversion of mechanical energy (flow) into heat rather than magnetic energy.

( ) Plasmas with low magnetic Reynolds number rotate in the wrong direction for magnetic field amplification to take place.

Exercise 1 - Magnetic buoyancy in the Sun

If the magnetic field amplitude of the considered flux tube is much greater than the one outside, as indicated in the problem set, the conservation of the total pressure (plasma plus magnetic) yields,

$$P + \frac{B^2}{2\mu_0} = P_{\rm ext} \; . \tag{1}$$

If we now use the assumption that the temperature variation between the interior and the exterior is negligible, we can write,

$$\frac{P}{\rho} \simeq \frac{P_{\rm ext}}{\rho_{\rm ext}} \simeq \frac{k_{\rm B}T}{\mu} \tag{2}$$

and use it in Eq. (1),

$$\Delta\rho = \rho_{\rm ext} - \rho \simeq \frac{B^2\mu}{2k_{\rm B}T\mu_0} \; . \tag{3}$$

Because of this density difference, the flux tube is subjected to an upward buoyancy force,

$$F_B = \Delta\rho V g \sim \frac{B^2a^2L}{2H\mu_0} \; , \tag{4}$$

where $H = k_{\rm B}T/(\mu g)$ is the characteristic pressure scale length and a^2L the flux tube volume.
Since the rising speed is assumed to be constant, this force has to be balanced by a drag force,

$$F_D \sim \frac{1}{2}\rho_{\rm ext}v^2aL \; , \tag{5}$$

where the drag coefficient was already set to 1 as indicated in the problem set. The cross-sectional area is aL. The balance of the two forces $F_D = F_B$ yields,

$$v \sim v_{\rm A}\left(\frac{a}{H}\right)^{1/2}\left(\frac{\rho}{\rho_{\rm ext}}\right)^{1/2} \; , \tag{6}$$

where $v_{\rm A} = \sqrt{B^2/(\mu_0\rho)}$ is the Alfven speed in the tube.

Exercise 2 - Kink instability of a flux rope / screw pinch

Let us start by considering solutions for the displacement of the form $\eta \sim e^{-i\omega t}$. The equation of motion provided in the problem set becomes,

$$-\omega^2\eta = v_{\rm A}^2[\eta'' + ik_0\eta'] \; . \tag{7}$$

We now look for a general solution of the type $\eta = C_1e^{ik_1z-i\omega t} + C_2e^{ik_2z-i\omega t}$. Imposing the boundary condition $\eta(z=0) = 0$, we find $C_1 = -C_2$,

$$\eta = C[e^{ik_1z} - e^{ik_2z}]e^{-i\omega t} \; . \tag{8}$$

Using the other boundary condition $\eta(z=L)=0$ we obtain,

$$0 = C[e^{ik_1L} - e^{ik_2L}] \quad \Rightarrow \quad 1 - e^{i(k_1-k_2)L} = 0 \quad \Rightarrow \quad k_1 - k_2 = \frac{2\pi n}{L} \; . \tag{9}$$

We can calculate the derivatives of η,

$$\eta' = iC(k_1 e^{ik_1z} - k_2 e^{ik_2z})e^{-i\omega t} \qquad \eta'' = -C(k_1^2 e^{ik_1z} - k_2^2 e^{ik_2z})e^{-i\omega t} \tag{10}$$

and substitute them in Eq. (7) together with Eq. (8) to obtain,

$$-\frac{\omega^2}{v_{\mathrm{A}}^2}(e^{ik_1z} - e^{ik_2z}) = -k_1^2 e^{ik_1z} + k_2^2 e^{ik_2z} - k_0k_1 e^{ik_1z} + k_0k_2 e^{ik_2z} =$$

$$= -(k_1^2 + k_0k_1)e^{ik_1z} + (k_2^2 + k_0k_2)e^{ik_2z} \; , \tag{11}$$

which requires

$$\frac{\omega^2}{v_{\mathrm{A}}^2} = k_1^2 + k_0k_1 = k_2^2 + k_0k_2 \; . \tag{12}$$

This can be rewritten as,

$$\frac{\omega^2}{v_{\mathrm{A}}^2} = \left(k_{1,2} + \frac{k_0}{2}\right)^2 - \frac{k_0^2}{4} \quad \Rightarrow \quad k_{1,2} = -\frac{k_0}{2} \pm \sqrt{\frac{\omega^2}{v_{\mathrm{A}}^2} + \frac{k_0^2}{4}} \; . \tag{13}$$

We can now obtain the following relation,

$$k_1 + \frac{k_0}{2} = \frac{k_1 - k_2}{2} \tag{14}$$

and insert it in Eq. (13),

$$\frac{\omega^2}{v_{\mathrm{A}}^2} = \left(\frac{k_1 - k_2}{2}\right)^2 - \frac{k_0^2}{4} \quad \Rightarrow \quad \frac{\omega^2}{v_{\mathrm{A}}^2} = \left(\frac{\pi n}{L}\right)^2 - \frac{k_0^2}{4} \; , \tag{15}$$

where we have used Eq. (9). We can now consider the most unstable mode with $n = 1$ and rewrite Eq. (15) as,

$$\omega^2 = \left(\frac{v_{\mathrm{A}}\pi}{L}\right)^2 \left[1 - \left(\frac{k_0 L}{2\pi}\right)^2\right] \underbrace{=}_{k_0 = B_\theta/(aB_z)} \left(\frac{v_{\mathrm{A}}\pi}{L}\right)^2 \left[1 - \left(\frac{B_\theta L}{2\pi a B_z}\right)^2\right] \; . \tag{16}$$

We should now notice that using the Ampere's law, the expression of the poloidal field B_θ at the border can be obtained,

$$\vec{\nabla} \times \vec{B}_\theta = \mu_0 \vec{J} \quad \Rightarrow \quad B_\theta = \frac{\mu_0 I}{2\pi a} \; , \tag{17}$$

where we have integrated on the poloidal cross section and used the *Stokes theorem*. Inserting this expression and the expression of $I_{\mathrm{K-S}}$ given in the problem set in Eq. (16), we finally find,

$$\omega^2 = \left(\frac{v_{\mathrm{A}}\pi}{L}\right)^2 \left[1 - \left(\frac{I}{I_{\mathrm{K-S}}}\right)^2\right] \; . \tag{18}$$

The growth rate of an instability is given by the imaginary part of ω, which start to be different from zero when the R.H.S. of Eq. (18) is smaller than zero ($[1-(I/I_{\mathrm{K-S}})^2] < 0$), i.e. when the plasma current exceeds the Kruskal-Shafranov critical current. In this case the growth rate is,

$$\gamma = \mathrm{Im}(\omega) = \frac{v_{\mathrm{A}}\pi}{L}\sqrt{\left(\frac{I}{I_{\mathrm{K-S}}}\right)^2 - 1}\,. \tag{19}$$