Plasma II - Exercises

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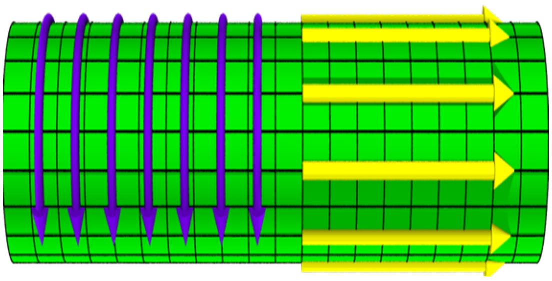
Examples of one-dimensional (1D) MHD equilibria are the cylindrical magnetic pinches, of which three varieties exist: the θ-pinch, the Z-pinch, and the screw-pinch, also known as the linear tokamak. In these three configurations, all equilibrium quantities, such as current, magnetic field and plasma pressure, only depend on the radial coordinate r.

Exercise 1 - The Bennett Z-pinch

In a Z-pinch the plasma is confined using a toroidal current and a poloidal magnetic field. Therefore, using cylindrical coordinates $\{\vec{e}_r, \vec{e}_\theta, \vec{e}_z\}$, we can set $\partial/\partial z = \partial/\partial\theta = 0$ for the Z-pinch.



Z-pinch: $\begin{cases} \vec{B} = (0, B_\theta(r), 0) \\ \vec{j} = (0, 0, j_z(r)) \end{cases}$

Figure 1: Schematic of a Z-pinch configuration with its $\vec{e}_\theta$-directed magnetic field (purple) and $\vec{e}_z$-directed plasma current (yellow).

Remember that in cylindrical coordinates the gradient, divergence and curl of a vector field $\vec{A} = (A_r(r,\theta,z), A_\theta(r,\theta,z), A_z(r,\theta,z))$ are, respectively:

$$\vec{\nabla} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right) \qquad \vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{A_r}{\partial\theta}\right)$$

a) Determine the general equilibrium configuration, i.e. find a general expression for the equilibrium relating the plasma pressure $p(r)$ to the toroidal current $j_z(r)$ or poloidal magnetic field $B_\theta(r)$.

b) Determine the radial profiles for the poloidal magnetic field $B_\theta(r)$ and the plasma pressure $p(r)$ for a Z-pinch with the radial current density profile,

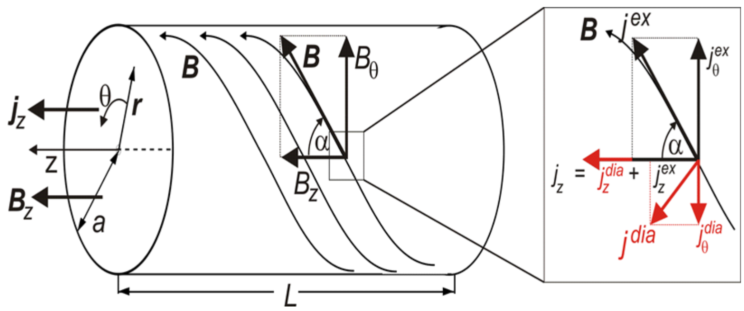
$$j_z(r) = \frac{2I}{\pi} \frac{a^2}{(r^2 + a^2)^2}$$

where a is the radius of the plasma column. Assume that the pressure decreases to zero at $r = a$.
Calculate the on-axis values (i.e. at $r = 0$) of j_z, B_θ and p using $I = 1\,\mathrm{kA}$ and $a = 1\,\mathrm{m}$. Draw qualitatively the radial profiles of these quantities.

c) (Optional - only, if you like to integrate) Compute the total thermal plasma energy W_{th} and the magnetic energy W_{mag}, assuming that this Z-pinch is built as a cylinder of finite length $L = 3\,\mathrm{m}$, terminated at both ends with appropriate electrodes so that the plasma cannot escape. What can you say on the MHD equilibrium of the Bennett pinch based on the ratio $W_{\mathrm{mag}}/W_{\mathrm{th}}$?

Exercise 2 - The screw-pinch

We can now combine the Z-pinch and the θ-pinch, to obtain a screw-pinch, which corresponds to a linear tokamak, i.e. a tokamak which is not folded over upon itself but which still requires electrodes at its ends to confine the plasma. In this configuration we externally impose the toroidal magnetic field $B_z(r)$, and we generate a toroidal current $j_z(r)$ to produce the poloidal magnetic field $B_\theta(r)$.



Screw-pinch:

$$\begin{cases} \vec{B} = (0, B_\theta(r), B_z(r)) \\ \vec{j} = (0, j_\theta(r), j_z(r)) \end{cases}$$

Figure 2: Magnetic configuration of a screw-pinch.

a) Determine the general equilibrium configuration for the screw-pinch.

b) If we impose a positive toroidal magnetic field $B_z(r) > 0$ and plasma current $j_z(r) > 0$, what are the conditions on the plasma pressure gradient to obtain a paramagnetic or diamagnetic behaviour? Think about j_θ and how it can affect B_z.

c) Determine expressions for the rotational transform and safety factor profiles of a screw-pinch.