Solid State Physics 4 — **Exercises session 13** — 25-05-2022

Exercises session 13

Abstract

The aim of this exercise is to derive the dispersion of spin waves in a simple cubic ferromagnetic (FM) and antiferromagnet (AF) compound. We will see that in a magnetic crystal, the spin wave dispersion is periodic in k-space with the period given by the reciprocal vector. Also, we will discuss the difference between the long-wavelength spin waves in FM and AF. The last exercise deals with diagonalizing the Heisenberg Hamiltonian and introduces the spin wave quantum (magnon).

Exercise 1: Semiclassical model of FM spin waves

a) Consider the Heisenberg Hamiltonian for a ferromagnet:

$$\hat{H} = -J\sum_{ij}\overrightarrow{S_i}\overrightarrow{S_j}, \tag{1}$$

and the corresponding equation of motion:

$$\frac{d\overrightarrow{S_n}}{dt} = i[\hat{H}, \overrightarrow{S_n}]. \tag{2}$$

Treating the spin operator as a vector, derive the equation of motion for its components S_i^x and S_i^y.

b) Working in the linear approximation, one can replace $S_i^z \approx S$ and $S_i^x, S_i^y << S_i^z$. Find the dispersion relation for FM spin waves assuming a plane wave solution of the form:

$$S_i^{x(y)} = A^{x(y)} e^{ikr_n - i\omega t} \tag{3}$$

c) Plot the dispersion relation for a cubic crystal with a lattice constant a.

Exercise 2: Semiclassical model of AF spin waves

a) We continue working within the semiclassical framework, but in this section, we consider an AF material. The simplest AF has two sublattices with the magnetic moments pointing in opposite directions (Fig. 1).

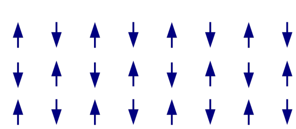

Figure 1: Schematics of an antiferromagnetically ordered material

Derive the equation of motion for the components of the spin vector for the up and down sublattices $S_i^x(y)$ for $i \in up$ and $S_i^x(y)$ for $i \in down$.

b) Replacing $S_i^z \approx S$ for $i \in up$ and $S_i^z \approx -S$ for $i \in down$ and considering:

$$S_i^{x(y)} = U^{x(y)} e^{ikr_n - i\omega t}, for i \in up,$$
$$S_i^{x(y)} = V^{x(y)} e^{ikr_n - i\omega t}, for i \in down, \tag{4}$$

find the dispersion relation for AF spin waves.

Exercise 3: Long-wavelength FM and AF spin waves

Discuss the difference of the FM and AF spin wave at small k.

Exercise 4 (bonus): Ferromagnetic magnons

a) To diagonalize the Hamiltonian in Eq. (1), one should introduce the creation and annihilation operators a and $a^\dagger$ satisfying the commutation relation $[a, a^\dagger] = 1$. Using Holstein-Primakoff transformations:

$$\begin{aligned} S^+ &= \sqrt{2S}\sqrt{1 - \frac{a^\dagger a}{2S}}\, a, \\ S^- &= \sqrt{2S}\, a^\dagger \sqrt{1 - \frac{a^\dagger a}{2S}}, \\ S^z &= S - a^\dagger a, \end{aligned} \tag{5}$$

where $S^{+(-)} = S^x + (-) i S^y$, show that the commutation relations for the spin algebra are consistent with the commutation relation for a and $a^\dagger$.

b) Diagonalize the Hamiltonian by substituting Eq. (5) in Eq. (1).

c) Compare the magnon dispersion with the dispersion obtained in Exercise 1.