15 Electroweak unification

One of the main goals of particle physics is to provide a unified picture of the fundamental particles and their interactions. In the nineteenth century, Maxwell provided a description of electricity and magnetism as different aspects of a unified electromagnetic theory. In the 1960s, Glashow, Salam and Weinberg (GSW) developed a unified picture of the electromagnetic and weak interactions. One consequence of the GSW electroweak model is the prediction of a weak neutral-current mediated by the neutral Z boson with well-defined properties. This short chapter describes electroweak unification and the properties of the W and Z bosons.

15.1 Properties of the W bosons

The W boson is a spin-1 particle with a mass of approximately 80 GeV. Its wave-function can be written in terms of a plane wave and a polarisation four-vector,

$$W^{\mu} = \epsilon_{\lambda}^{\mu} e^{-ip\cdot x} = \epsilon_{\lambda}^{\mu} e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}.$$

For a massive spin-1 particle the polarisation four-vector ϵ_{λ}^{μ} is restricted to one of three possible polarisation states (see Appendix D). For a W boson travelling in the z-direction, the three orthogonal polarisation states λ can be written as

$$\epsilon_{-}^{\mu} = \frac{1}{\sqrt{2}}(0,1,-i,0), \quad \epsilon_{L}^{\mu} = \frac{1}{m_{\mathrm{W}}}(p_z,0,0,E) \quad \text{and} \quad \epsilon_{+}^{\mu} = -\frac{1}{\sqrt{2}}(0,1,i,0). \qquad (15.1)$$

These states represent two transverse polarisation modes $\epsilon_{\pm}$, corresponding to circularly polarised spin-1 states with $S_z = \pm 1$, and a longitudinal $S_z = 0$ state.

15.1.1 W-boson decay

The calculation of the W-boson decay rate provides a good illustration of the use of polarisation four-vectors in matrix element calculations. The lowest-order Feynman diagram for the $\mathrm{W}^- \to \mathrm{e}^-\overline{\nu}_{\mathrm{e}}$ decay is shown in Figure 15.1. The matrix element for the decay is obtained using the appropriate Feynman rules. The final-state electron and antineutrino are written respectively as the adjoint particle spinor

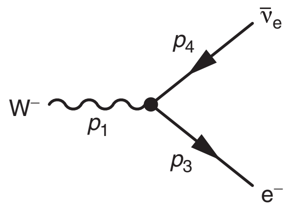

Fig. 15.1 The lowest-order Feynman diagram for $W^- \to e^-\overline{\nu}_e$.

$\overline{u}(p_3)$ and the antiparticle spinor $v(p_4)$. The initial-state W^- is written as $\epsilon^\lambda_\mu(p_1)$, where λ indicates one of the three possible polarisation states. Finally, the vertex factor for the weak charged-current is the usual $V - A$ interaction

$$-i\frac{g_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5).$$

Using these Feynman rules, the matrix element for $W^- \to e^-\overline{\nu}_e$ is given by

$$-i\mathcal{M}_{fi} = \epsilon^\lambda_\mu(p_1)\,\overline{u}(p_3)\left[-i\frac{g_W}{\sqrt{2}}\gamma^\mu\frac{1}{2}(1-\gamma^5)\right]v(p_4),$$

and therefore

$$\mathcal{M}_{fi} = \frac{g_W}{\sqrt{2}}\,\epsilon^\lambda_\mu(p_1)\,\overline{u}(p_3)\gamma^\mu\frac{1}{2}(1-\gamma^5)v(p_4). \tag{15.2}$$

This expression can be written as the four-vector scalar product of the W-boson four-vector polarisation and the lepton current,

$$\mathcal{M}_{fi} = \frac{g_W}{\sqrt{2}}\,\epsilon^\lambda_\mu(p_1)j^\mu, \tag{15.3}$$

where the leptonic weak charged-current j^μ is given by

$$j^\mu = \overline{u}(p_3)\gamma^\mu\frac{1}{2}(1-\gamma^5)v(p_4). \tag{15.4}$$

It is convenient to consider the $W^- \to e^-\overline{\nu}_e$ decay in the rest frame of the W boson, as illustrated in Figure 15.2. Given that $m_W \gg m_e$, the mass of the electron can be neglected and the four-vectors of the W^-, e^- and $\overline{\nu}_e$ can be taken to be

$$\begin{aligned}
p_1 &= (m_W, 0, 0, 0),\\
p_3 &= (E, E\sin\theta, 0, E\cos\theta),\\
p_4 &= (E, -E\sin\theta, 0, -E\cos\theta),
\end{aligned}$$

with $E = m_W/2$. In the ultra-relativistic limit, where the helicity states are the same as the chiral states, only left-handed helicity particle states and right-handed helicity antiparticle states contribute to the weak interaction. In this case, the leptonic current of (15.4) can be written

$$j^\mu = \overline{u}_\downarrow(p_3)\gamma^\mu v_\uparrow(p_4), \tag{15.5}$$

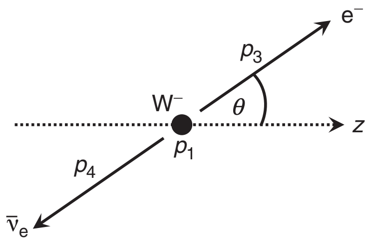

Fig. 15.2 The decay $W^- \to e^-\overline{\nu}_e$ in the W-boson rest frame.

where $\overline{u}_\downarrow(p_3)$ and $v_\uparrow(p_4)$ are respectively the left-handed particle and right-handed antiparticle helicity spinors for the electron and electron antineutrino. The leptonic current of (15.5) is identical to that encountered for the $\mu^+\mu^-$ current in the s-channel process $e^+e^- \to \mu^+\mu^-$ and is given by (6.17) with $E = m_W/2$,

$$j^\mu = m_W(0, -\cos\theta, -i, \sin\theta).$$

For a W boson at rest, the three possible polarisation states of (15.1) are

$$\epsilon_-^\mu = \tfrac{1}{\sqrt{2}}(0, 1, -i, 0), \quad \epsilon_L^\mu = (0, 0, 0, 1) \quad \text{and} \quad \epsilon_+^\mu = -\tfrac{1}{\sqrt{2}}(0, 1, i, 0).$$

Therefore, from (15.3), the matrix elements for the decay $W^- \to e^-\overline{\nu}_e$ in the three possible W-boson polarisation states are

$$\begin{aligned}
\mathcal{M}_- &= \tfrac{g_W m_W}{2}(0, 1, -i, 0)\cdot(0, -\cos\theta, -i, \sin\theta) = \tfrac{1}{2}g_W m_W(1 + \cos\theta),\\
\mathcal{M}_L &= \tfrac{g_W m_W}{\sqrt{2}}(0, 0, 0, 1)\cdot m_W(0, -\cos\theta, -i, \sin\theta) = -\tfrac{1}{\sqrt{2}}g_W m_W \sin\theta,\\
\mathcal{M}_+ &= -\tfrac{g_W m_W}{2}(0, 1, i, 0)\cdot m_W(0, -\cos\theta, -i, \sin\theta) = \tfrac{1}{2}g_W m_W(1 - \cos\theta).
\end{aligned}$$

Hence, for the three possible W-boson polarisations

$$\begin{aligned}
|\mathcal{M}_-|^2 &= g_W^2 m_W^2 \tfrac{1}{4}(1 + \cos\theta)^2,\\
|\mathcal{M}_L|^2 &= g_W^2 m_W^2 \tfrac{1}{2}\sin^2\theta,\\
|\mathcal{M}_+|^2 &= g_W^2 m_W^2 \tfrac{1}{4}(1 - \cos\theta)^2.
\end{aligned}$$

The resulting angular distributions of the decay products for each of the different W-boson polarisations can be understood by noting that the LH and RH helicities of the electron and antineutrino imply that they are produced in a spin-1 state aligned with the direction of the neutrino, as shown in Figure 15.3. The angular

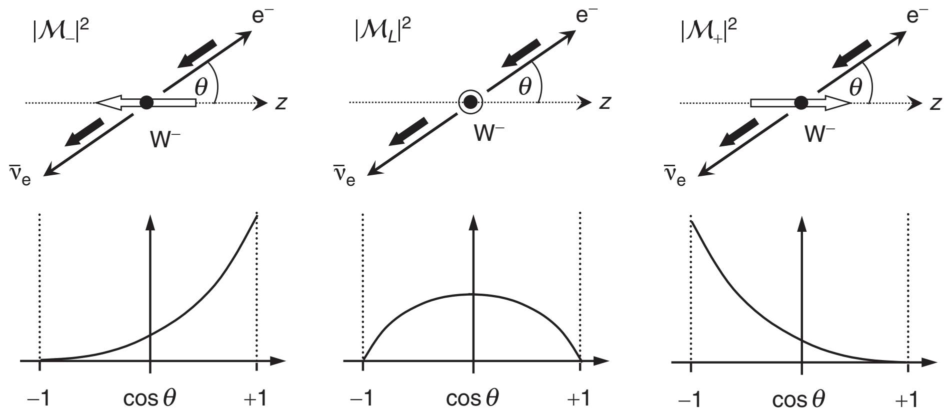

Fig. 15.3 The angular distributions of the electron and electron antineutrino in the decay $W^- \to e^-\overline{\nu}_e$ for the three possible W-boson polarisation states.

distributions then follow from the quantum mechanical properties of spin-1, as discussed in Section 6.3.

The total decay rate is determined by the spin-averaged matrix element squared, which (for unpolarised W decays) is given by

$$\begin{aligned}\left\langle|\mathcal{M}_{fi}|^2\right\rangle &= \tfrac{1}{3}\left(|\mathcal{M}_-|^2 + |\mathcal{M}_L|^2 + |\mathcal{M}_+|^2\right)\\ &= \tfrac{1}{3}g_W^2 m_W^2\left[\tfrac{1}{4}(1+\cos\theta)^2 + \tfrac{1}{2}\sin^2\theta + \tfrac{1}{4}(1-\cos\theta)^2\right]\\ &= \tfrac{1}{3}g_W^2 m_W^2. \end{aligned} \tag{15.6}$$

Hence, after averaging over the three polarisation states of the W boson, there is no preferred direction for the final-state particles that are, as expected, produced isotropically in the W-boson rest frame. The $W^- \to e^-\overline{\nu}_e$ decay rate is obtained by substituting the expression for the spin-averaged matrix element of (15.6) into the decay rate formula of (3.49),

$$\Gamma = \frac{p^*}{32\pi^2 m_W^2}\int \left\langle|\mathcal{M}_{fi}|^2\right\rangle d\Omega^* = \frac{p^*}{8\pi m_W^2}\left\langle|\mathcal{M}_{fi}|^2\right\rangle,$$

where p^* is the momentum of the electron (or antineutrino) in the centre-of-mass frame. If the masses of the final-state particles are neglected, $p^* = m_W/2$, and therefore the $W^- \to e^-\overline{\nu}_e$ decay rate is given by

$$\Gamma(W^- \to e^-\overline{\nu}_e) = \frac{g_W^2 m_W}{48\pi}. \tag{15.7}$$

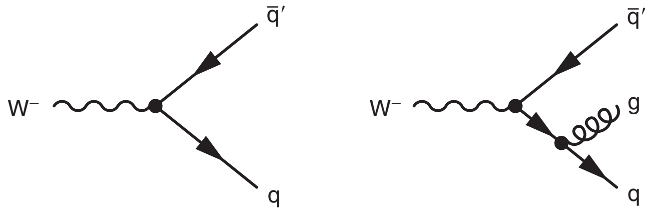

Fig. 15.4 The lowest-order Feynman diagram for $W^- \to q\overline{q}'$ and the first-order QCD correction from $W^- \to q\overline{q}'g$.

The expression of (15.7) gives the partial decay width for $W^- \to e^-\overline{\nu}_e$. To calculate the total decay rate of the W boson, all possible decay modes have to be considered. From the lepton universality of the weak charged-current (and neglecting the very small differences due to the lepton masses), the three leptonic decay modes have the same partial decay rates,

$$\Gamma(W^- \to e^-\overline{\nu}_e) = \Gamma(W^- \to \mu^-\overline{\nu}_\mu) = \Gamma(W^- \to \tau^-\overline{\nu}_\tau).$$

The W boson also can decay to all flavours of quarks with the exception of the top quark, which is too massive ($m_t > m_W$). The decay rate of the W boson to a particular quark flavour needs to account for the elements of the CKM matrix and the three possible colours of the final-state quarks, therefore the decay rates relative to $\Gamma_{e\nu} = \Gamma(W^- \to e^-\overline{\nu}_e)$ are

$$\begin{aligned}
\Gamma(W^- \to d\overline{u}) &= 3|V_{ud}|^2\,\Gamma_{e\nu}, \quad & \Gamma(W^- \to d\overline{c}) &= 3|V_{cd}|^2\,\Gamma_{e\nu},\\
\Gamma(W^- \to s\overline{u}) &= 3|V_{us}|^2\,\Gamma_{e\nu}, \quad & \Gamma(W^- \to s\overline{c}) &= 3|V_{cs}|^2\,\Gamma_{e\nu},\\
\Gamma(W^- \to b\overline{u}) &= 3|V_{ub}|^2\,\Gamma_{e\nu}, \quad & \Gamma(W^- \to b\overline{c}) &= 3|V_{cb}|^2\,\Gamma_{e\nu}.
\end{aligned}$$

From the unitarity of the CKM matrix,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \quad \text{and} \quad |V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1,$$

and the lowest-order prediction for the W-boson decay rate to quarks is

$$\Gamma(W^- \to q\overline{q}') = 6\,\Gamma(W^- \to e^-\overline{\nu}_e).$$

In addition to the lowest-order $W \to q\overline{q}'$ process, the QCD correction from the process $W \to q\overline{q}'g$, shown in Figure 15.4, enhances the decay rate to hadronic final states by a factor

$$\kappa_{QCD} = \left[1 + \frac{\alpha_S(m_W)}{\pi}\right] \approx 1.038. \tag{15.8}$$

Thus the total decay rate of the W boson to either quarks or to the three possible leptonic final states is

$$\Gamma_W = (3 + 6\,\kappa_{QCD})\,\Gamma(W^- \to e^-\overline{\nu}_e) \approx 9.2 \times \frac{g_W^2 m_W}{48\pi} = 2.1\,\text{GeV},$$

and the branching ratio of the W boson to hadronic final states is

$$BR(W \to q\overline{q}') = \frac{6\,\kappa_{QCD}}{3 + 6\,\kappa_{QCD}} = 67.5\%. \tag{15.9}$$

The prediction of $\Gamma_W = 2.1\,\text{GeV}$ is in good agreement with the measured value of $\Gamma_W = 2.085 \pm 0.042\,\text{GeV}$ (see Chapter 16). Because the mass of the W boson is large, so is the total decay width, and the lifetime of the W boson is only $\mathcal{O}(10^{-25}\,\text{s})$.

15.1.2 W-pair production

The fact that the force carrying particles of the weak interaction possess the charge of the electromagnetic interaction is already suggestive that the weak and electromagnetic forces are somehow related. Further hints of electroweak unification are provided by the observation that the coupling constants of the electromagnetic and weak interactions are of the same order of magnitude (see Section 11.5.1). However, there are also strong theoretical arguments for why a theory with just the weak charged current must be incomplete.

Pairs of W bosons can be produced in e^+e^- annihilation at an electron–positron collider or in $q\overline{q}$ annihilation at a hadron collider. The three lowest-order Feynman diagrams for the process $e^+e^- \to W^+W^-$ are shown in Figure 15.5. The t-channel neutrino exchange diagram represents a purely weak charged-current process. The s-channel photon exchange diagram is an electromagnetic process, which arises because the W^+ and W^- carry electromagnetic charge. With the first two diagrams of Figure 15.5 alone, the calculated $e^+e^- \to W^+W^-$ cross section is found to increase with centre-of-mass energy without limit, as shown in Figure 15.6.

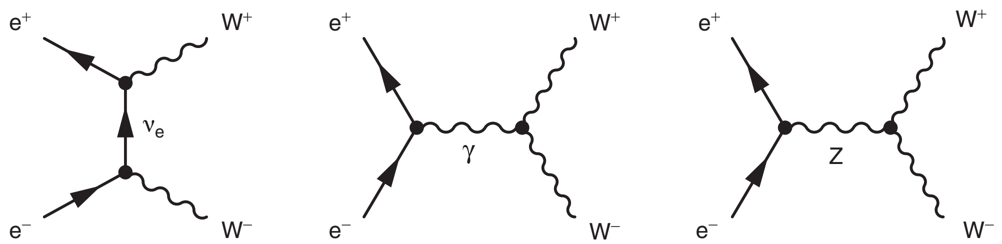

Fig. 15.5 The three lowest-order Feynman diagram for $e^+e^- \to W^+W^-$.

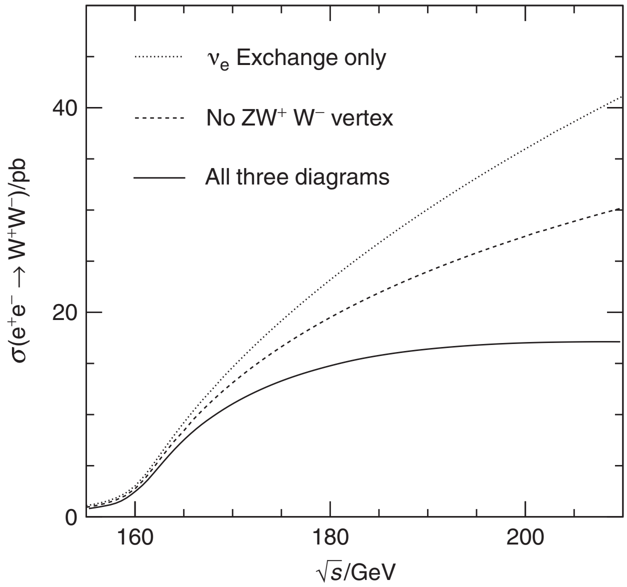

Fig. 15.6 The predicted $e^+e^- \to W^+W^-$ production cross section for three cases: only the ν_e-exchange diagram; the ν_e-exchange and γ-exchange diagrams; and all three Feynman diagrams of Figure 15.5.

At some relatively high centre-of-mass energy, the cross section violates quantum mechanical unitarity, whereby particle probability is no longer conserved; the calculated number of W-pairs produced in the interaction exceeds the incident e^+e^- flux. This problematic high-energy behaviour of the $e^+e^- \to W^+W^-$ cross section indicates that the theory with just the first two diagrams of Figure 15.5 is incomplete. Because the *s*- and *t*-channel diagrams interfere negatively, the problem would be even worse with the neutrino exchange diagram alone,

$$|\mathcal{M}_\nu + \mathcal{M}_\gamma|^2 < |\mathcal{M}_\nu|^2.$$

The problem of unitarity violation in $e^+e^- \to W^+W^-$ production is resolved naturally in the electroweak theory, which predicts an additional gauge boson, the neutral Z. Because the contribution to the $e^+e^- \to W^+W^-$ cross section from the Z-exchange diagram interferes negatively,

$$|\mathcal{M}_\nu + \mathcal{M}_\gamma + \mathcal{M}_Z|^2 < |\mathcal{M}_\nu + \mathcal{M}_\gamma|^2,$$

the calculated $e^+e^- \to W^+W^-$ cross section is well behaved at all centre-of-mass energies, as shown in Figure 15.6. This partial cancellation only works because the couplings of the γ, $W^\pm$ and the new Z boson are related to each other in the unified electroweak model.

15.2 The weak interaction gauge group

In Section 10.1 it was shown that QED and QCD are associated with respective U(1) and SU(3) local gauge symmetries. The charged-current weak interaction is associated with invariance under SU(2) local phase transformations,

$$\varphi(x) \to \varphi'(x) = \exp\left[ig_W\, \boldsymbol{\alpha}(x) \cdot \mathbf{T}\right] \varphi(x). \tag{15.10}$$

Here $\mathbf{T}$ are the three generators of the SU(2) group that can be written in terms of the Pauli spin matrices,

$$\mathbf{T} = \tfrac{1}{2}\boldsymbol{\sigma},$$

and $\boldsymbol{\alpha}(x)$ are three functions which specify the local phase at each point in space-time. The required local gauge invariance can only be satisfied by the introduction of three gauge fields, W_μ^k with $k = 1, 2, 3$, corresponding to three gauge bosons $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$. Because the generators of the SU(2) gauge transformation are the 2×2 Pauli spin-matrices, the wavefunction $\varphi(x)$ in (15.10) must be written in terms of two components. In analogy with the definition of isospin, $\varphi(x)$ is termed a weak isospin doublet. Since the weak charged-current interaction associated with the $W^\pm$ couples together different fermions, the weak isospin doublets must contain flavours differing by one unit of electric charge, for example

$$\varphi(x) = \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}.$$

In this weak isospin space, the ν_e and e^- have total weak isospin $I_W = \frac{1}{2}$ and third component of weak isospin $I_W^{(3)}(\nu_e) = +\frac{1}{2}$ and $I_W^{(3)}(e^-) = -\frac{1}{2}$. Since the observed form of the weak charged-current interaction couples only to left-handed chiral particle states and right-handed chiral antiparticle states, the gauge transformation of (15.10) can affect only LH particles and RH antiparticles. To achieve this, RH particle and LH antiparticle chiral states are placed in weak isospin singlets with $I_W = 0$ and are therefore unaffected by the SU(2) local gauge transformation. The weak isospin doublets are composed only of LH chiral particle states and RH chiral antiparticle states and, for this reason, the symmetry group of the weak interaction is referred to as $SU(2)_L$.

The weak isospin doublets are constructed from the weak eigenstates and therefore account for the mixing in the CKM and PMNS matrices. For example, the u quark appears in a doublet with the weak eigenstate d′, as defined in (14.3). The upper member of the doublet, with $I_W^{(3)} = +1/2$, is always the particle state which differs by plus one unit in electric charge relative to the lower member of the doublet,

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \quad \begin{pmatrix} u \\ d' \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L.$$

This common ordering within the doublets is necessary for a consistent definition of the physical $W^\pm$ bosons. The right-handed particle chiral states are placed in weak isospin singlets with $I_W = I_W^{(3)} = 0$,

$$e_R^-, \quad \mu_R^-, \quad \tau_R^-, \quad u_R, \quad c_R, \quad t_R, \quad d_R, \quad s_R, \quad b_R.$$

Because the weak isospin singlets are unaffected by the $SU(2)_L$ local gauge transformation of the weak interaction, they do not couple to the gauge bosons of the symmetry.

The requirement of local gauge invariance implies the modification of the Dirac equation to include a new interaction term, analogous to (10.11),

$$ig_W T_k \gamma^\mu W_\mu^k \varphi_L = ig_W \tfrac{1}{2}\sigma_k \gamma^\mu W_\mu^k \varphi_L, \tag{15.11}$$

where φ_L represents a weak isospin doublet of left-handed chiral particles. This form of the interaction gives rise to three weak currents, one for each of the three gauge fields W^k. In the case of the weak isospin doublet formed from the left-handed electron and the electron neutrino,

$$\varphi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix},$$

the three weak currents, one for each of the Pauli spin-matrices, are

$$j_1^\mu = \frac{g_W}{2}\,\overline{\varphi}_L \gamma^\mu \sigma_1 \varphi_L, \quad j_2^\mu = \frac{g_W}{2}\,\overline{\varphi}_L \gamma^\mu \sigma_2 \varphi_L \quad \text{and} \quad j_3^\mu = \frac{g_W}{2}\,\overline{\varphi}_L \gamma^\mu \sigma_3 \varphi_L,$$

where $\overline{\varphi}_L = \begin{pmatrix} \overline{\nu}_L & \overline{e}_L \end{pmatrix}$ contains the left-handed chiral adjoint spinors, $\overline{\nu}_L$ and $\overline{e}_L$. The weak charged-currents are related to the weak isospin raising and lowering operators, $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, which step between the two states within a weak isospin doublet. The four-vector currents corresponding to the exchange of the physical $W^\pm$ bosons are

$$\begin{aligned} j_\pm^\mu = \tfrac{1}{\sqrt{2}}\left(j_1^\mu \pm i j_2^\mu\right) &= \frac{g_W}{\sqrt{2}}\,\overline{\varphi}_L \gamma^\mu \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)\varphi_L, \\ &= \frac{g_W}{\sqrt{2}}\,\overline{\varphi}_L \gamma^\mu \sigma_\pm \varphi_L. \end{aligned}$$

The physical W bosons can be identified as the linear combinations

$$W_\mu^\pm = \tfrac{1}{\sqrt{2}}\left(W_\mu^{(1)} \mp iW_\mu^{(2)}\right), \tag{15.12}$$

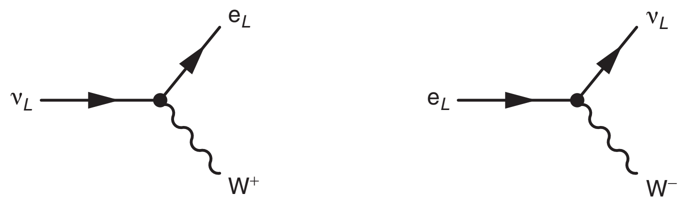

Fig. 15.7 The weak charged-current interaction vertices between the e^- and the ν_e weak eigenstate.

such that the weak currents can be written

$$\mathbf{j}^\mu \cdot \mathbf{W}_\mu = j_1^\mu W_\mu^{(1)} + j_2^\mu W_\mu^{(2)} + j_3^\mu W_\mu^{(3)} \equiv j_+^\mu W_\mu^+ + j_-^\mu W_\mu^- + j_3^\mu W_\mu^{(3)}.$$

The current j_+^μ, which corresponds to the exchange of a W^+ boson, can be expressed as

$$\begin{aligned} j_+^\mu &= \frac{g_W}{\sqrt{2}}\overline{\varphi}_L\gamma^\mu\sigma_+\varphi_L = \frac{g_W}{\sqrt{2}}\begin{pmatrix}\overline{\nu}_L & \overline{e}_L\end{pmatrix}\gamma^\mu\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\\ &= \frac{g_W}{\sqrt{2}}\overline{\nu}_L\gamma^\mu e_L \equiv \frac{g_W}{\sqrt{2}}\overline{\nu}\,\gamma^\mu\tfrac{1}{2}(1-\gamma^5)e. \end{aligned}$$

Similarly, the current corresponding to the W^- vertex is

$$\begin{aligned} j_-^\mu &= \frac{g_W}{\sqrt{2}}\overline{\varphi}_L\gamma^\mu\sigma_-\varphi_L = \frac{g_W}{\sqrt{2}}\begin{pmatrix}\overline{\nu}_L & \overline{e}_L\end{pmatrix}\gamma^\mu\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\\ &= \frac{g_W}{\sqrt{2}}\overline{e}_L\gamma^\mu\nu_L \equiv \frac{g_W}{\sqrt{2}}\overline{e}\,\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\nu. \end{aligned}$$

Thus, the $SU(2)_L$ symmetry of the weak interaction results in the now familiar weak charged-currents

$$j_+^\mu = \frac{g_W}{\sqrt{2}}\overline{\nu}\,\gamma^\mu\tfrac{1}{2}(1-\gamma^5)e \quad\text{and}\quad j_-^\mu = \frac{g_W}{\sqrt{2}}\overline{e}\,\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\nu,$$

corresponding to the W^+ and W^- vertices shown in Figure 15.7.

In addition to the two weak charged-currents, j_+ and j_-, which arise from linear combinations of the $W^{(1)}$ and $W^{(2)}$, the $SU(2)_L$ gauge symmetry implies the existence a weak neutral-current given by

$$j_3^\mu = g_W\overline{\varphi}_L\gamma^\mu\tfrac{1}{2}\sigma_3\varphi_L.$$

The weak neutral-current, written in terms of the component fermions, is

$$\begin{aligned} j_3^\mu &= g_W\tfrac{1}{2}\begin{pmatrix}\overline{\nu}_L & \overline{e}_L\end{pmatrix}\gamma^\mu\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}\\ &= g_W\tfrac{1}{2}\overline{\nu}_L\gamma^\mu\nu_L - g_W\tfrac{1}{2}\overline{e}_L\gamma^\mu e_L. \end{aligned} \tag{15.13}$$

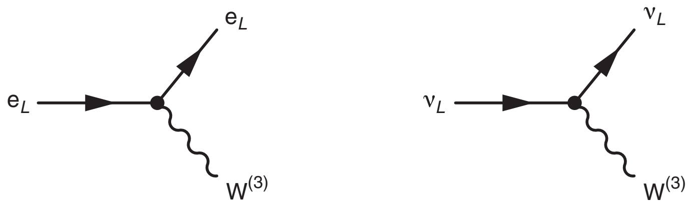

Fig. 15.8 The weak neutral-current interaction of the e^- and ν_e.

Hence the $SU(2)_L$ symmetry of the weak interaction implies the existence of the weak neutral-current corresponding to the vertices shown in Figure 15.8, with

$$j_3^\mu = I_W^{(3)} g_W \bar{f} \gamma^\mu \tfrac{1}{2}(1 - \gamma^5) f, \tag{15.14}$$

where f denotes the fermion spinor. The sign in this expression is determined by the third component of weak isospin $I_W^{(3)} = \pm 1/2$. Because RH particles/LH antiparticles have $I_W^{(3)} = 0$, they do not couple to the weak neutral-current corresponding to the $W^{(3)}$.

15.3 Electroweak unification

It is tempting to identify the neutral-current of (15.14) as that due to the exchange of the Z boson, in which case the Z boson would correspond to the $W^{(3)}$ of the $SU(2)_L$ local gauge symmetry. This would imply that the weak neutral-current coupled only to left-handed particles and right-handed antiparticles. This is in contradiction with experiment, which shows that the physical Z boson couples to both left- and right-handed chiral states (although not equally).

Of the four observed bosons of QED and the weak interaction, the photon and the Z boson, with the corresponding fields A_μ and Z_μ, are both neutral. Consequently, it is plausible that they can be expressed in terms of quantum state formed from two neutral bosons, one of which is the $W^{(3)}$ associated with the $SU(2)_L$ local gauge symmetry. In the electroweak model of Glashow, Salam and Weinberg (GSW), the U(1) gauge symmetry of electromagnetism is replaced with a new $U(1)_Y$ local gauge symmetry

$$\psi(x) \to \psi'(x) = \hat{U}(x)\psi(x) = \exp\left[ig'\frac{Y}{2}\zeta(x)\right]\psi(x), \tag{15.15}$$

giving rise to a new gauge field B_μ that couples to a new kind of charge, termed weak hypercharge Y. The resulting interaction term is

$$g'\frac{Y}{2}\gamma^\mu B_\mu \psi, \tag{15.16}$$

which has the same form as the interaction from the U(1) symmetry of QED,

$$Qe\gamma^{\mu}A_{\mu}\psi,$$

with Qe replaced by $Yg'/2$. In the unified electroweak model, the photon and Z boson are written as linear combinations of the B_{μ} and neutral $W_{\mu}^{(3)}$ of the weak interaction,

$$A_{\mu} = +B_{\mu}\cos\theta_{W} + W_{\mu}^{(3)}\sin\theta_{W}, \tag{15.17}$$

$$Z_{\mu} = -B_{\mu}\sin\theta_{W} + W_{\mu}^{(3)}\cos\theta_{W}, \tag{15.18}$$

where θ_W is the weak mixing angle. This mixing of the neutral fields of the $U(1)_Y$ and $SU(2)_L$ gauge symmetries might seem contrived; however, it arises naturally in the Higgs mechanism (see Chapter 17). From (15.17) and (15.18), the physical currents of QED and the weak neutral current are

$$j_{em}^{\mu} = j_{Y}^{\mu}\cos\theta_{W} + j_{3}^{\mu}\sin\theta_{W}, \tag{15.19}$$

$$j_{Z}^{\mu} = -j_{Y}^{\mu}\sin\theta_{W} + j_{3}^{\mu}\cos\theta_{W}. \tag{15.20}$$

The GSW model of electroweak unification implies that the couplings of the weak and electromagnetic interactions are related. This can be seen by considering the interactions of the electron and the electron neutrino. The weak neutral-current associated with the $W^{(3)}$ is given by (15.13) and involves only left-handed particles,

$$j_{3}^{\mu} = \tfrac{1}{2}g_{W}\,\overline{\nu}_{L}\gamma^{\mu}\nu_{L} - \tfrac{1}{2}g_{W}\,\overline{e}_{L}\gamma^{\mu}e_{L}. \tag{15.21}$$

The currents from the interaction term of (15.16), which treats left- and right-handed states equally, are

$$j_{Y}^{\mu} = \tfrac{1}{2}g'Y_{e_L}\,\overline{e}_{L}\gamma^{\mu}e_{L} + \tfrac{1}{2}g'Y_{e_R}\,\overline{e}_{R}\gamma^{\mu}e_{R} + \tfrac{1}{2}g'Y_{\nu_L}\,\overline{\nu}_{L}\gamma^{\mu}\nu_{L} + \tfrac{1}{2}g'Y_{\nu_R}\,\overline{\nu}_{R}\gamma^{\mu}\nu_{R}, \tag{15.22}$$

where, for example, Y_{e_L} is the weak hypercharge of the left-handed electron. The current for the electromagnetic interaction, written in terms of the chiral components of the electron, is

$$j_{em}^{\mu} = Q_{e}e\,\overline{e}_{L}\gamma^{\mu}e_{L} + Q_{e}e\,\overline{e}_{R}\gamma^{\mu}e_{R}.$$

Since the neutrino is a neutral particle its electromagnetic current is zero. For the GSW model to work, it must reproduce the observed couplings of QED. From (15.19) the electromagnetic current can be written

$$j_{em}^{\mu} = Q_{e}e\,\overline{e}_{L}\gamma^{\mu}e_{L} + Q_{e}e\,\overline{e}_{R}\gamma^{\mu}e_{R} = j_{Y}^{\mu}\cos\theta_{W} + j_{3}^{\mu}\sin\theta_{W},$$

where j_Y^μ and j_3^μ, which include terms for the electron neutrino, are given by (15.21) and (15.22). Hence the terms in electromagnetic current j_{em}^μ, including those for the neutrinos which are zero, can be equated to

$$\bar{e}_L\gamma^\mu e_L : \qquad Q_e e = \tfrac{1}{2}g' Y_{e_L}\cos\theta_W - \tfrac{1}{2}g_W\sin\theta_W, \tag{15.23}$$

$$\bar{\nu}_L\gamma^\mu \nu_L : \qquad 0 = \tfrac{1}{2}g' Y_{\nu_L}\cos\theta_W + \tfrac{1}{2}g_W\sin\theta_W, \tag{15.24}$$

$$\bar{e}_R\gamma^\mu e_R : \qquad Q_e e = \tfrac{1}{2}g' Y_{e_R}\cos\theta_W, \tag{15.25}$$

$$\bar{\nu}_R\gamma^\mu \nu_R : \qquad 0 = \tfrac{1}{2}g' Y_{\nu_R}\cos\theta_W. \tag{15.26}$$

Equations (15.23)–(15.26) relate the couplings of electromagnetism to those of the weak interaction and the couplings associated with the $U(1)_Y$ symmetry.

In the GSW model, the underlying gauge symmetry of the electroweak sector of the Standard Model is the $U(1)_Y$ of weak hypercharge and the $SU(2)_L$ of the weak interaction, written as $U(1)_Y \times SU(2)_L$. For invariance under $U(1)_Y$ *and* $SU(2)_L$ local gauge transformations, the weak hypercharges of the particles in a weak isospin doublet must be the same, for example $Y_{e_L} = Y_{\nu_L}$. If this were not the case, a $U(1)_Y$ local gauge transformation would introduce a phase difference between the two components of a weak isospin doublet, breaking the $SU(2)_L$ symmetry. The weak hypercharge assignments of the fermions can be expressed as a linear combination of the electromagnetic charge Q and the third component of weak isospin $I_W^{(3)}$,

$$Y = \alpha Q + \beta I_W^{(3)}.$$

The charges and third component of weak isospin for the left-handed electron and the left-handed electron neutrino are respectively $\left(Q = -1, I_W^{(3)} = -\tfrac{1}{2}\right)$ and $\left(Q = 0, I_W^{(3)} = +\tfrac{1}{2}\right)$, and therefore

$$Y_{\nu_L} = +\tfrac{1}{2}\beta \quad \text{and} \quad Y_{e_L} = -\alpha - \tfrac{1}{2}\beta.$$

From the requirement that $Y_{e_L} = Y_{\nu_L}$, it follows that $\beta = -\alpha$ and the weak hypercharge can be identified as

$$Y = 2\left(Q - I_W^{(3)}\right). \tag{15.27}$$

The factor of two in (15.27) is purely conventional; a different choice could be absorbed into the definition of g' without modifying the actual couplings. The weak hypercharges of the e_L and ν_L are therefore

$$Y_{e_L} = Y_{\nu_L} = -1.$$

Since $Y_{e_L} = Y_{\nu_L}$, subtracting (15.23) from (15.24) gives the relationship between the weak and electromagnetic couplings in terms of the weak mixing angle,

$$e = g_W \sin\theta_W. \tag{15.28}$$

The sum of (15.23) and (15.24) gives

$$Q_e e = \tfrac{1}{2} g' (Y_{e_L} + Y_{\nu_L}) \cos\theta_W .$$

Since $Q_{e_L} = -1$ and $Y_{e_L} = Y_{\nu_L} = -1$, the coupling g' is related to the electron charge by

$$e = g' \cos\theta_W . \tag{15.29}$$

Finally, from (15.27), the weak hypercharge assignments of the $I_W^{(3)} = 0$ right-handed states are

$$Y_{e_R} = -2 \quad \text{and} \quad Y_{\nu_R} = 0,$$

which when substituted into (15.25) and (15.26) give the correct electromagnetic charges of $Q = -1$ and $Q = 0$ for the e_R and ν_R.

The unified electroweak model is able to provide a consistent picture of the electromagnetic interactions of the fermions with the relation,

$$e = g_W \sin\theta_W = g' \cos\theta_W , \tag{15.30}$$

and where the weak hypercharge is given by

$$Y = 2\left(Q - I_W^{(3)}\right).$$

The weak mixing angle has been measured in a number of different ways, including the studies of $e^+e^- \to Z \to f\bar{f}$, described in Chapter 16. The average of the measurements of $\sin^2\theta_W$ gives

$$\sin^2\theta_W = 0.23146 \pm 0.00012. \tag{15.31}$$

From (15.28) and the measured value of $\sin^2\theta_W$, the expected ratio of the weak to electromagnetic coupling constants is

$$\frac{\alpha}{\alpha_W} = \frac{e^2}{g_W^2} = \sin^2\theta_W \sim 0.23,$$

consistent with the measured values discussed previously in Section 11.5.1.

15.3.1 The couplings of the Z

At this point it might be tempting to think that the procedure for electroweak unification has just replaced two independent couplings, e and g_W, by a single unified coupling and the weak mixing angle. However, once the couplings in the electroweak model are chosen to reproduce the observed electromagnetic couplings,

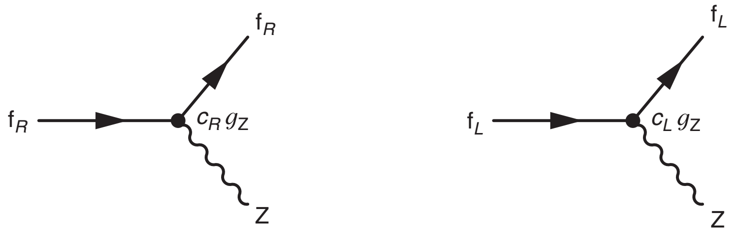

Fig. 15.9 The weak neutral-current interaction vertices for the physical Z boson and the chiral states of a fermion f.

the couplings of the Z boson to all the fermions are completely specified. The current from the interaction between the Z boson and a fermion of flavour f is given by (15.20),

$$j_Z^\mu = -\tfrac{1}{2}g' \sin\theta_W [Y_{f_L}\overline{u}_L\gamma^\mu u_L + Y_{f_R}\overline{u}_R\gamma^\mu u_R] + I_W^{(3)} g_W \cos\theta_W [\overline{u}_L\gamma^\mu u_L],$$

where $u_{L/R}$ and $\overline{u}_{L/R}$ are the spinors and adjoints spinors for LH and RH chiral states. Using (15.27) to express the weak hypercharge in terms of Q and $I_W^{(3)}$ implies

$$j_Z^\mu = -g' \sin\theta_W \left[\left(Q_f - I_W^{(3)}\right)\overline{u}_L\gamma^\mu u_L + Q_f\overline{u}_R\gamma^\mu u_R\right] + I_W^{(3)} g_W \cos\theta_W [\overline{u}_L\gamma^\mu u_L].$$

Collecting up the factors in front of the left- and right-handed currents gives

$$j_Z^\mu = \left[-g'\left(Q_f - I_W^{(3)}\right)\sin\theta_W + I_W^{(3)} g_W \cos\theta_W\right]\overline{u}_L\gamma^\mu u_L - [g'\sin\theta_W Q_f]\,\overline{u}_R\gamma^\mu u_R.$$

From (15.30) it can be seen that $g' = g_W \tan\theta_W$ and therefore

$$j_Z^\mu = g_W\left[-\left(Q_f - I_W^{(3)}\right)\frac{\sin^2\theta_W}{\cos\theta_W} + I_W^{(3)}\cos\theta_W\right]\overline{u}_L\gamma^\mu u_L - g_W\left[\frac{\sin^2\theta_W}{\cos\theta_W}Q_f\right]\overline{u}_R\gamma^\mu u_R. \tag{15.32}$$

Defining the coupling to the physical Z boson as

$$g_Z = \frac{g_W}{\cos\theta_W} \equiv \frac{e}{\sin\theta_W\cos\theta_W},$$

allows the neutral-current due to the Z boson to be written as

$$j_Z^\mu = g_Z\left(I_W^{(3)} - Q_f\sin^2\theta_W\right)\overline{u}_L\gamma^\mu u_L - g_Z\left(Q_f\sin^2\theta_W\right)\overline{u}_R\gamma^\mu u_R.$$

Hence the couplings of the Z boson to left- and right-handed chiral states, shown in Figure 15.9, are

$$j_Z^\mu = g_Z\,(c_L\,\overline{u}_L\gamma^\mu u_L + c_R\overline{u}_R\gamma^\mu u_R), \tag{15.33}$$

with

$$c_L = I_W^{(3)} - Q_f\sin^2\theta_W \quad \text{and} \quad c_R = -Q_f\sin^2\theta_W. \tag{15.34}$$

Thus, the Z boson couples to both left- and right-handed chiral states, but not equally. This should come as no surprise; the current associated with the Z boson

Table 15.1 The charge, $I_W^{(3)}$ and weak hypercharge assignments of the fundamental fermions and their couplings to the Z assuming $\sin^2\theta_W = 0.231\,46$.

fermion	Q_f	$I_W^{(3)}$	Y_L	Y_R	c_L	c_R	c_V	c_A
ν_e, ν_μ, ν_τ	0	$+\frac{1}{2}$	−1	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
e^-, μ^-, τ^-	−1	$-\frac{1}{2}$	−1	−2	−0.27	+0.23	−0.04	$-\frac{1}{2}$
u, c, t	$+\frac{2}{3}$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{4}{3}$	+0.35	−0.15	+0.19	$+\frac{1}{2}$
d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{2}{3}$	−0.42	+0.08	−0.35	$-\frac{1}{2}$

has contributions from the weak interaction, which couples only to left-handed particles, and from the B_μ field associated with the $U(1)_Y$ local gauge symmetry, which treats left- and right-handed states equally.

The couplings of the Z boson to fermions also can be expressed in terms of vector and axial-vector couplings using the chiral projection operators of (6.33),

$$\overline{u}_L\gamma^\mu u_L = \overline{u}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u \quad \text{and} \quad \overline{u}_R\gamma^\mu u_R = \overline{u}\gamma^\mu \tfrac{1}{2}(1+\gamma^5)u,$$

such that the current j_Z^μ of (15.33) becomes

$$\begin{aligned} j_Z^\mu &= g_Z\overline{u}\gamma^\mu \left[c_L\tfrac{1}{2}(1-\gamma^5) + c_R\tfrac{1}{2}(1+\gamma^5)\right]u \\ &= g_Z\overline{u}\gamma^\mu\, \tfrac{1}{2}\left[(c_L+c_R)-(c_L-c_R)\gamma^5\right]u. \end{aligned}$$

Therefore the weak neutral-current can be written as

$$j_Z^\mu = \tfrac{1}{2}g_Z\overline{u}\left(c_V\gamma^\mu - c_A\gamma^\mu\gamma^5\right)u, \tag{15.35}$$

where the vector and axial-vector couplings of the Z boson are

$$c_V = (c_L + c_R) = I_W^{(3)} - 2Q\sin^2\theta_W, \tag{15.36}$$

$$c_A = (c_L - c_R) = I_W^{(3)}. \tag{15.37}$$

In terms of these vector and axial-vector couplings, the Feynman rule associated with the Z-boson interaction vertex is

$$-i\tfrac{1}{2}g_Z\gamma^\mu\left[c_V - c_A\gamma^5\right]. \tag{15.38}$$

Because the weak neutral-current contains both vector and axial-vector couplings, it does not conserve parity (see Section 11.3); this also immediately follows from its different couplings to left- and right-handed chiral states.

In the Standard Model, once $\sin^2\theta_W$ is known, the couplings of the Z boson to the fermions are predicted exactly. For $\sin^2\theta_W = 0.23146$, the predicted couplings of the fermions to the Z boson are listed in Table 15.1, both in terms of the vector and axial-vector couplings (c_V, c_A) and the couplings to left- and right-handed chiral states, (c_L, c_R).

15.4 Decays of the Z

The calculation of the Z-boson total decay width and branching ratios follows closely that for the decay of the W boson, given in Section 15.1.1. However, whereas the W boson couples only to left-handed chiral particle states, the Z boson couples to both left- and right-handed states. Nevertheless, because the weak neutral-current is a vector/axial-vector interaction, the currents due to certain chiral combinations are still zero. For example, for the decay $Z \to f\bar{f}$, the weak neutral-current where both the fermion and antifermion are right-handed is zero, which can be seen from

$$\begin{aligned}\bar{u}_R\gamma^\mu(c_V - c_A\gamma^5)v_R &= u^\dagger\tfrac{1}{2}(1+\gamma^5)\gamma^0\gamma^\mu(c_V - c_A\gamma^5)\tfrac{1}{2}(1-\gamma^5)v \\ &= \tfrac{1}{4}u^\dagger\gamma^0(1-\gamma^5)(1+\gamma^5)\gamma^\mu(c_V - c_A\gamma^5)v \\ &= \tfrac{1}{4}\bar{u}\gamma^\mu P_L P_R(c_V - c_A\gamma^5)v = 0.\end{aligned}$$

Consequently, in the limit where the masses of the fermions in the decay $Z \to f\bar{f}$ can be neglected, only the two helicity combinations shown in Figure 15.10 give non-zero matrix elements for the decay of the Z boson.

The Z-boson decay rate either can be calculated from first principles (see Problem 15.3) or can be obtained from the spin-averaged matrix element of (15.6), derived previously for W-boson decay. For the helicity combination where the decay of the Z boson gives a LH particle and RH antiparticle, the spin-averaged matrix element is the same as that for W-boson decay, but with

$$\tfrac{1}{2}g_W^2 \to g_Z^2 c_L^2, \quad \Rightarrow \quad \langle|\mathcal{M}_L|^2\rangle = \tfrac{2}{3}c_L^2 g_Z^2 m_Z^2.$$

The corresponding matrix element for the Z decay to a RH particle and LH antiparticle will be proportional to c_R rather than c_L. After averaging over the spin states and decay angle, all other factors will be the same. Therefore the spin-averaged matrix element squared for $Z \to f\bar{f}$ is

$$\langle|\mathcal{M}|^2\rangle = \langle|\mathcal{M}_L|^2 + |\mathcal{M}_R|^2\rangle = \tfrac{2}{3}(c_L^2 + c_R^2)g_Z^2 m_Z^2. \qquad (15.39)$$

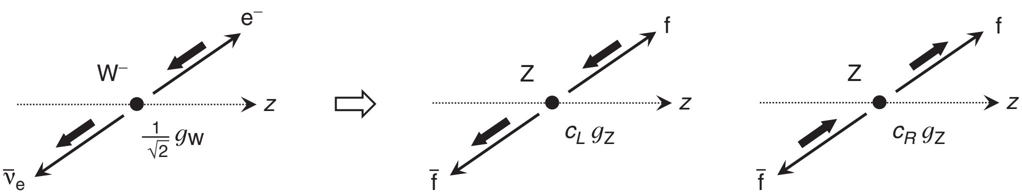

Fig. 15.10 The possible helicities in the decays $W^- \to e^-\bar{\nu}_e$ and $Z \to f\bar{f}$.

This can be expressed in terms of the vector and axial-vector couplings of the Z boson using $c_V = c_L + c_R$ and $c_A = c_L - c_R$, which implies that $c_V^2 + c_A^2 = 2(c_L^2 + c_R^2)$. Hence

$$\langle|\mathcal{M}|^2\rangle = \tfrac{1}{3}(c_V^2 + c_A^2)g_Z^2 m_Z^2, \tag{15.40}$$

from which it follows that

$$\Gamma(\mathrm{Z} \to \mathrm{f}\bar{\mathrm{f}}) = \frac{g_Z^2 m_Z}{48\pi}(c_V^2 + c_A^2). \tag{15.41}$$

15.4.1 Z width and branching ratios

The Z-boson partial decay rates depend on g_Z and m_Z. The measured value of mass of the Z boson is $m_Z = 91.2\,\mathrm{GeV}$ (see Section 16.2.1). The numerical value of g_Z can be obtained from the measured values of the Fermi constant and $\sin^2\theta_W$,

$$g_Z^2 = \frac{g_W^2}{\cos^2\theta_W} = \frac{8m_W^2}{\sqrt{2}\cos^2\theta_W}G_F \approx 0.55.$$

The partial decay rate to a particular fermion flavour can be calculated from (15.41) using the appropriate vector and axial-vector couplings. For example, in the case of the decay $\mathrm{Z} \to \nu_e\bar{\nu}_e$, the neutrino vector and axial-vector couplings are $c_V = c_A = \frac{1}{2}$, and therefore

$$\Gamma(\mathrm{Z} \to \nu_e\bar{\nu}_e) = \frac{g_Z^2 m_Z}{48\pi}\left(\frac{1}{4} + \frac{1}{4}\right) = 167\,\mathrm{MeV}. \tag{15.42}$$

Because the Z boson couples to all fermions, it can decay to all flavours with the exception of the top quark ($m_t > m_Z$). The total decay width Γ_Z is given by the sum of the partial decay widths

$$\Gamma_Z = \sum_f \Gamma(\mathrm{Z} \to \mathrm{f}\bar{\mathrm{f}}).$$

The Z-boson couplings, listed in Table 15.1, are the same for all three generations, and thus the total decay width can be written

$$\Gamma_Z = 3\,\Gamma(\mathrm{Z} \to \nu_e\bar{\nu}_e) + 3\,\Gamma(\mathrm{Z} \to e^+e^-) + 3 \times 2\,\Gamma(\mathrm{Z} \to u\bar{u}) + 3 \times 3\,\Gamma(\mathrm{Z} \to d\bar{d}),$$

where the additional factors of three multiplying the decays to quarks account for colour, and only two decays to up-type quarks are possible since $m_t > m_Z$. Using the couplings in Table 15.1, and multiplying the hadronic decay widths by $[1 + \alpha_S(Q^2)/\pi]$ to account for the gluon radiation in the decay, the total decay width of the Z is predicted to be

$$\Gamma_Z \approx 2.5\,\mathrm{GeV},$$

and the branching ratios of the Z boson, given by $Br(\mathrm{Z} \to \mathrm{f\bar{f}}) = \Gamma(\mathrm{Z} \to \mathrm{f\bar{f}})/\Gamma_\mathrm{Z}$, are

$$\begin{aligned} Br(\mathrm{Z} \to \nu_e\bar{\nu}_e) = Br(\mathrm{Z} \to \nu_\mu\bar{\nu}_\mu) = Br(\mathrm{Z} \to \nu_\tau\bar{\nu}_\tau) &\approx 6.9\%, \\ Br(\mathrm{Z} \to e^+e^-) = Br(\mathrm{Z} \to \mu^+\mu^-) = Br(\mathrm{Z} \to \tau^+\tau^-) &\approx 3.5\%, \\ Br(\mathrm{Z} \to \mathrm{u\bar{u}}) = Br(\mathrm{Z} \to \mathrm{c\bar{c}}) &\approx 12\%, \\ Br(\mathrm{Z} \to \mathrm{d\bar{d}}) = Br(\mathrm{Z} \to \mathrm{s\bar{s}}) = Br(\mathrm{Z} \to \mathrm{b\bar{b}}) &\approx 15\%. \end{aligned}$$

Grouping together the decays to neutrinos, charged leptons, and quarks gives

$$Br(\mathrm{Z} \to \nu\bar{\nu}) \approx 21\%, \quad Br(\mathrm{Z} \to \ell^+\ell^-) \approx 10\% \ \text{ and } \ Br(\mathrm{Z} \to \text{hadrons}) \approx 69\%,$$

and thus almost 70% of Z decays result in final states with jets.

Summary

In the Standard Model, the weak charged-current is associated with an $SU(2)_L$ local gauge symmetry. This gives rise to the W^+ and W^- bosons and a neutral gauge field, $W^{(3)}$. In the GSW model of electroweak unification, this neutral field mixes with a photon-like field of the $U(1)_Y$ gauge symmetry to give the physical photon and Z-boson fields

$$\begin{aligned} A_\mu &= +B_\mu \cos\theta_W + W_\mu^{(3)} \sin\theta_W \\ Z_\mu &= -B_\mu \sin\theta_W + W_\mu^{(3)} \cos\theta_W, \end{aligned}$$

where θ_W is the weak mixing angle. Within this unified model, the couplings of the γ, W and Z are related by

$$e = g_W \sin\theta_W = g_Z \sin\theta_W \cos\theta_W.$$

Within the unified electroweak model, once θ_W is known, the properties of the Z boson are completely specified. The precise tests of these predictions are main the subject of the next chapter.

Problems

15.1 Draw all possible lowest-order Feynman diagrams for the processes:

$$e^+e^- \to \mu^+\mu^-, \quad e^+e^- \to \nu_\mu\bar{\nu}_\mu, \quad \nu_\mu e^- \to \nu_\mu e^- \quad \text{and} \quad \bar{\nu}_e e^- \to \bar{\nu}_e e^-.$$

15.2 Draw the lowest-order Feynman diagram for the decay $\pi^0 \to \nu_\mu\bar{\nu}_\mu$ and explain why this decay is effectively forbidden.

15.3 Starting from the matrix element, work through the calculation of the $Z \to f\bar{f}$ partial decay rate, expressing the answer in terms of the vector and axial-vector couplings of Z. Taking $\sin^2\theta_W = 0.2315$, show that

$$R_\mu = \frac{\Gamma(Z \to \mu^+\mu^-)}{\Gamma(Z \to \text{hadrons})} \approx \frac{1}{20}.$$

15.4 Consider the purely neutral-current (NC) process $\nu_\mu e^- \to \nu_\mu e^-$.

(a) Show that in the limit where the electron mass can be neglected, the spin-averaged matrix element for $\nu_\mu e^- \to \nu_\mu e^-$ can be written

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{2}\left(\left|\mathcal{M}_{LL}^{NC}\right|^2 + \left|\mathcal{M}_{LR}^{NC}\right|^2\right),$$

where

$$\mathcal{M}_{LL}^{NC} = 2c_L^{(\nu)}c_L^{(e)}\frac{g_Z^2 s}{m_Z^2} \quad \text{and} \quad \mathcal{M}_{RR}^{NC} = 2c_L^{(\nu)}c_R^{(e)}\frac{g_Z^2 s}{m_Z^2}\tfrac{1}{2}(1+\cos\theta^*),$$

and θ^* is the angle between the directions of the incoming and scattered neutrino in the centre-of-mass frame.

(b) Hence find an expression for the $\nu_\mu e^-$ neutral-current cross section in terms of the laboratory frame neutrino energy.

15.5 The two lowest-order Feynman diagrams for $\nu_e e^- \to \nu_e e^-$ are shown in Figure 13.5. Because both diagrams produce the same final state, the amplitudes have to be added before the matrix element is squared. The matrix element for the charged-current (CC) process is

$$\mathcal{M}_{LL}^{CC} = \frac{g_W^2 s}{m_W^2}.$$

(a) In the limit where the lepton masses and the q^2 term in the W-boson propagator can be neglected, write down expressions for spin-averaged matrix elements for the processes

$$\nu_\mu e^- \to \nu_\mu e^-, \quad \nu_e e^- \to \nu_e e^- \quad \text{and} \quad \nu_\mu e^- \to \nu_e \mu^-.$$

(b) Using the relation $g_Z/m_Z = g_W/m_W$, show that

$$\sigma(\nu_\mu e^- \to \nu_\mu e^-) : \sigma(\nu_e e^- \to \nu_e e^-) : \sigma(\nu_\mu e^- \to \nu_e \mu^-) = c_L^2 + \tfrac{1}{3}c_R^2 : (1+c_L)^2 + \tfrac{1}{3}c_R^2 : 1,$$

where c_L and c_R refer to the couplings of the left- and right-handed charged leptons to the Z.

(c) Find numerical values for these ratios of NC + CC : NC : CC cross sections and comment on the sign of the interference between the NC and CC diagrams.