11 The weak interaction

This chapter provides an introduction to the weak interaction, which is mediated by the massive W^+ and W^- bosons. The main topics covered are: the origin of parity violation; the *V–A* form of the interaction vertex; and the connection to Fermi theory, which is the effective low-energy description of the weak charged current. The calculation of the decay rate of the charged pion is used to illustrate the rôle of helicity in weak decays. The purpose of this chapter is to describe the overall structure of the weak interaction; the applications are described in the following chapters on charged-current interactions, neutrino oscillations and CP violation in the weak decays of neutral mesons.

11.1 The weak charged-current interaction

At the fundamental level, QED and QCD share a number of common features. Both interactions are mediated by massless neutral spin-1 bosons and the spinor part of the QED and QCD interaction vertices have the same $\overline{u}(p')\gamma^{\mu}u(p)$ form. The *charged-current* weak interaction differs in almost all respects. It is mediated by massive charged $W^{\pm}$ bosons and consequently couples together fermions differing by one unit of electric charge. It is also the only place in the Standard Model where parity is not conserved. The parity violating nature of the interaction can be directly related to the form of the interaction vertex, which differs from that of QED and QCD.

11.2 Parity

The parity operation is equivalent to spatial inversion through the origin, $\mathbf{x} \to -\mathbf{x}$. In general, in quantum mechanics the parity transformation can be associated with the operator $\hat{P}$, defined by

$$\psi(\mathbf{x}, t) \to \psi'(\mathbf{x}, t) = \hat{P}\psi(\mathbf{x}, t) = \psi(-\mathbf{x}, t).$$

The original wavefunction is clearly recovered if the parity operator is applied twice,

$$\hat{P}\hat{P}\psi(\mathbf{x},t) = \hat{P}\psi(-\mathbf{x},t) = \psi(\mathbf{x},t),$$

and hence the parity operator is its own inverse,

$$\hat{P}\hat{P} = I. \tag{11.1}$$

If physics is invariant under parity transformations, then the parity operation must be unitary

$$\hat{P}^{\dagger}\hat{P} = I. \tag{11.2}$$

From (11.1) and (11.2), it can be inferred that

$$\hat{P}^{\dagger} = \hat{P},$$

and therefore $\hat{P}$ is a Hermitian operator that corresponds to an observable property of a quantum-mechanical system. Furthermore, *if* the interaction Hamiltonian commutes with $\hat{P}$, parity is an observable *conserved* quantity in the interaction. In this case, if $\psi(\mathbf{x},t)$ is an eigenstate of the Hamiltonian, it is also an eigenstate of the parity operator with an eigenvalue P,

$$\hat{P}\psi(\mathbf{x},t) = P\psi(\mathbf{x},t).$$

Acting on this eigenvalue equation with $\hat{P}$ gives

$$\hat{P}\hat{P}\psi(\mathbf{x},t) = P\hat{P}\psi(\mathbf{x},t) = P^2\psi(\mathbf{x},t),$$

which implies that $P^2 = 1$ since $\hat{P}\hat{P} = I$. Because $\hat{P}$ is Hermitian, its eigenvalues are real and are therefore equal to ± 1.

11.2.1 Intrinsic parity

Fundamental particles, despite being point-like, possess an intrinsic parity. In Section 4.9, it was shown that the parity operator for Dirac spinors is γ^0, which in the Dirac–Pauli matrix representation is

$$\hat{P} = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It was also shown that spin-half *particles*, which necessarily satisfy the Dirac equation, have the opposite parity to the corresponding *antiparticles*. By convention, the particle states are defined to have positive intrinsic parity; for example $P(\mathrm{e}^-) = P(\nu_\mathrm{e}) = P(\mathrm{q}) = +1$, and therefore antiparticles have negative intrinsic parity, for example $P(e^+) = P(\overline{\nu}_e) = P(\overline{q}) = -1$. From the Quantum Field Theory describing the force carrying particles, it can be shown that the vector bosons responsible for the electromagnetic, strong and weak forces all have negative intrinsic parity,

$$P(\gamma) = P(g) = P(W^\pm) = P(Z) = -1.$$

11.2.2 Parity conservation in QED

Parity conservation in QED arises naturally from the form of the interaction. For example, the matrix element for the QED process of $e^-q \to e^-q$ scattering, shown in Figure 11.1, can be written as the four-vector scalar product

$$\mathcal{M} = \frac{Q_q e^2}{q^2} j_e \cdot j_q,$$

where the electron and quark currents are defined by

$$j_e^\mu = \overline{u}(p_3)\gamma^\mu u(p_1) \quad \text{and} \quad j_q^\nu = \overline{u}(p_4)\gamma^\nu u(p_2). \tag{11.3}$$

The equivalent matrix element for the parity transformed process, where the three-momenta of all the particles are reversed, can be obtained by applying the parity operator $\hat{P} = \gamma^0$ to the spinors of (11.3). Since Dirac spinors transform as

$$u \xrightarrow{\hat{P}} \hat{P}u = \gamma^0 u, \tag{11.4}$$

the adjoint spinors transform as

$$\overline{u} = u^\dagger\gamma^0 \xrightarrow{\hat{P}} (\hat{P}u)^\dagger\gamma^0 = u^\dagger\gamma^{0\dagger}\gamma^0 = u^\dagger\gamma^0\gamma^0 = \overline{u}\gamma^0,$$

and hence

$$\overline{u} \xrightarrow{\hat{P}} \overline{u}\gamma^0. \tag{11.5}$$

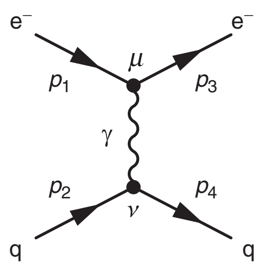

Fig. 11.1 The lowest-order Feynman diagram for the QED *t*-channel electron–quark scattering process.

From (11.4) and (11.5), it can be seen that the four-vector currents of (11.3) become

$$j_e^\mu = \overline{u}(p_3)\gamma^\mu u(p_1) \xrightarrow{\hat{P}} \overline{u}(p_3)\gamma^0\gamma^\mu\gamma^0 u(p_1).$$

Because $\gamma^0\gamma^0 = I$, the time-like component of the current is unchanged by the parity operation,

$$j_e^0 \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^0\gamma^0 u = \overline{u}\gamma^0 u = j_e^0.$$

The space-like components of j^μ, with indices $k = 1, 2, 3$, transform as

$$j_e^k \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^k\gamma^0 u = -\overline{u}\gamma^k\gamma^0\gamma^0 u = -\overline{u}\gamma^k u = -j_e^k,$$

since $\gamma^0\gamma^k = -\gamma^k\gamma^0$. Therefore, as expected, the parity operation changes the signs of the space-like components of the four-vector current but the time-like component remains unchanged. Consequently, the four-vector scalar product in the QED matrix element, $j_e \cdot j_q = j_e^0 j_q^0 - j_e^k j_q^k$, transforms to

$$j_e \cdot j_q = j_e^0 j_q^0 - j_e^k j_q^k \xrightarrow{\hat{P}} j_e^0 j_q^0 - (-j_e^k)(-j_q^k) = j_e \cdot j_q, \tag{11.6}$$

and it can be concluded that the QED matrix element is invariant under the parity operation. Hence the terms in the Hamiltonian related to the QED interaction are invariant under parity transformations. This invariance implies that

parity is conserved in QED.

Apart from the colour factors, the QCD interaction has the same form as QED and consequently

parity is conserved in QCD.

The conservation of parity in strong and electromagnetic interactions needs to be taken into account when considering particle decays. For example, consider the two decays

$$\rho^0(1^-) \rightarrow \pi^+(0^-) + \pi^-(0^-) \quad \text{and} \quad \eta(0^-) \rightarrow \pi^+(0^-) + \pi^-(0^-),$$

where the J^P values are shown in brackets. The total parity of the two-body final state is the product of the intrinsic parities of the particles and the parity of the orbital wavefunction, which is given by $(-1)^\ell$, where ℓ is the orbital angular momentum in the final state. In order to conserve angular momentum, the π^+ and π^- in the $\rho^0 \rightarrow \pi^+\pi^-$ decay are produced with relative orbital angular moment

$\ell = 1$, whereas the $\pi^+\pi^-$ in the decay of the η must have $\ell = 0$. Therefore, conservation of parity in the two decays can be expressed as follows:

$$P(\rho^0) = P(\pi^+)\cdot P(\pi^-)\cdot(-1)^{\ell=1} \quad \Rightarrow \quad -1 = (-1)(-1)(-1) \quad \checkmark$$
$$P(\eta) = P(\pi^+)\cdot P(\pi^-)\cdot(-1)^{\ell=0} \quad \Rightarrow \quad -1 = (-1)(-1)(+1) \quad \times$$

Hence the strong interaction decay process $\rho^0 \to \pi^+\pi^-$ is allowed, but the strong decay $\eta \to \pi^+\pi^-$ does not occur as it would violate the conservation of parity that is implicit in the strong interaction Hamiltonian. It can also be shown that the QED and QCD interactions are invariant under the charge conjugation operation $\hat{C}$, defined in Section 4.7.5, which changes particles into antiparticles and vice versa, and therefore there is a corresponding conserved quantity $C = \pm 1$.

Scalars, pseudoscalars, vectors and axial vectors

Physical quantities can be classified according to their rank (dimensionality) and parity inversion properties. For example, single-valued *scalar* quantities, such as mass and temperature, are invariant under parity transformations. *Vector* quantities, such as position and momentum, change sign under parity transformations, $\mathbf{x} \to -\mathbf{x}$ and $\mathbf{p} \to -\mathbf{p}$. There is also a second class of vector quantity, known as an *axial vector*, which is sometimes referred to as a pseudovector. Axial vectors are formed from the cross product of two vector quantities, and therefore do not change sign under parity transformations. One example is angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$. Because both $\mathbf{x}$ and $\mathbf{p}$ change sign under parity, the axial vector $\mathbf{L}$ is unchanged. Other examples of axial vectors include the magnetic moment and the magnetic flux density $\mathbf{B}$, which is related to the current density $\mathbf{j}$ by the Biot–Savart law, $\mathrm{d}\mathbf{B} \propto \mathbf{j} \times \mathrm{d}^3\mathbf{x}$. Scalar quantities can be formed out of scalar products of two vectors or two axial vectors, the simplest example being the magnitude squared of the momentum vector, $\mathrm{p}^2 = \mathbf{p}\cdot\mathbf{p}$. There is a second class of scalar quantity known as a *pseudoscalar*. Pseudoscalars are single-valued quantities formed from the product of a vector and an axial vector, and consequently change sign under the parity operation. One important example of a pseudoscalar is helicity, $h \propto \mathbf{S}\cdot\mathbf{p}$. The different scalar and vector quantities are listed in Table 11.1.

Table 11.1 The parity properties of scalars, pseudoscalars, vectors and axial vectors.

	Rank	Parity	Example
Scalar	0	+	Temperature, T
Pseudoscalar	0	−	Helicity, h
Vector	1	−	Momentum, $\mathbf{p}$
Axial vector	1	+	Angular momentum, $\mathbf{L}$

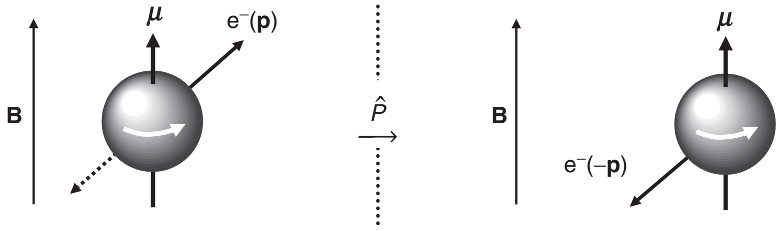

Fig. 11.2 The β-decay of polarised ^{60}Co. On the left, an electron is emitted in a particular direction. On the right the parity inverted equivalent is shown.

11.2.3 Parity violation in nuclear β-decay

The parity inversion properties of the different types of physical quantity can be exploited to investigate whether parity is conserved in the weak interaction. In 1957, Wu and collaborators studied nuclear β-decay of polarised cobalt-60,

$$^{60}\mathrm{Co} \to {}^{60}\mathrm{Ni}^* + \mathrm{e}^- + \overline{\nu}_\mathrm{e}.$$

The ^{60}Co nuclei, which possess a permanent nuclear magnetic moment $\boldsymbol{\mu}$, were aligned in a strong magnetic field **B** and the β-decay electrons were detected at different polar angles with respect to this axis, as shown in Figure 11.2. Because both **B** and $\boldsymbol{\mu}$ are axial vectors, they do not change sign under the parity transformation. Hence when viewed in the parity inverted "mirror", the only quantity that changes sign is the vector momentum of the emitted electron. Hence, if parity were conserved in the weak interaction, the rate at which electrons were emitted at a certain direction relative to the **B**-field would be identical to the rate in the opposite direction. Experimentally, it was observed that more electrons were emitted in the hemisphere opposite to the direction of the applied magnetic field than in the hemisphere in the direction of the applied field, thus providing a clear demonstration that

parity is NOT conserved in the weak interaction.

From this observation it can be concluded that, unlike QED and QCD, the weak interaction does not have four-vector currents of the form $j^\mu = \overline{u}(p')\gamma^\mu u(p)$.

11.3 *V* − *A* structure of the weak interaction

QED and QCD are vector interactions with a current of the form $j^\mu = \overline{u}(p')\gamma^\mu u(p)$. This particular combination of spinors and γ-matrices transforms as a four vector (as shown in Appendix B.3). From the observation of parity violation, the

Table 11.2 Lorentz-invariant bilinear covariant currents.

Type	Form	Components	Boson spin
Scalar	$\overline{\psi}\phi$	1	0
Pseudoscalar	$\overline{\psi}\gamma^5\phi$	1	0
Vector	$\overline{\psi}\gamma^\mu\phi$	4	1
Axial vector	$\overline{\psi}\gamma^\mu\gamma^5\phi$	4	1
Tensor	$\overline{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\phi$	6	2

weak interaction vertex is required have a different form. However, the requirement of Lorentz invariance of the interaction matrix element severely restricts the possible forms of the interaction. The general bilinear combination of two spinors can be written $\overline{u}(p')\Gamma u(p)$, where Γ is a 4×4 matrix formed from products of the Dirac γ-matrices. It turns out that there are only five combinations of individual γ-matrices that have the correct Lorentz transformation properties, such that they can be combined into a Lorentz-invariant matrix element. These combinations are called *bilinear covariants* and give rise to the possible *scalar*, *pseudoscalar*, *vector*, *axial vector* and *tensor* currents listed in Table 11.2.

In QED, the factor $g_{\mu\nu}$ in the matrix element arises from the sum over the $(2J + 1) + 1$ polarisation states of the $J^P = 1^-$ virtual photon, which includes the time-like component of the polarisation four-vector. These four polarisation states correspond to the four degrees of freedom of the vector current $j^\mu = \overline{\psi}\gamma^\mu\phi$, labelled by the index $\mu = 0, 1, 2, 3$. The single component scalar and pseudoscalar interactions therefore can be associated with the exchange of a spin-0 boson ($J = 0$), which possesses just a single degree of freedom. Similarly, the six non-zero components of a tensor interaction can be associated with the exchange of a spin-2 boson ($J = 2$), with $(2J + 1) + 1 = 6$ polarisation states for the spin-2 virtual particle.

The most general Lorentz-invariant form for the interaction between a fermion and a boson is a linear combination of the bilinear covariants. If this is restricted to the exchange of a spin-1 (vector) boson, the most general form for the interaction is a linear combination of vector and axial vector currents,

$$j^\mu \propto \overline{u}(p')(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)u(p) = g_V j^\mu_V + g_A j^\mu_A,$$

where g_V and g_A are vector and axial vector coupling constants and the current has been decomposed into vector and axial vector components

$$j^\mu_V = \overline{u}(p')\gamma^\mu u(p) \quad \text{and} \quad j^\mu_A = \overline{u}(p')\gamma^\mu\gamma^5 u(p).$$

The parity transformation properties of j_V^μ were derived in Section 11.2.2. The parity transformation properties of the pure axial vector current can be obtained the same way

$$j_A^\mu = \overline{u}\gamma^\mu\gamma^5 u \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^\mu\gamma^5\gamma^0 u = -\overline{u}\gamma^0\gamma^\mu\gamma^0\gamma^5 u,$$

which follows from $\gamma^5\gamma^0 = -\gamma^0\gamma^5$. Hence the time-like component of the axial vector current transforms as

$$j_A^0 \xrightarrow{\hat{P}} -\overline{u}\gamma^0\gamma^0\gamma^0\gamma^5 u = -\overline{u}\gamma^0\gamma^5 u = -j_A^0,$$

and the space-like components transform as

$$j_A^k \xrightarrow{\hat{P}} -\overline{u}\gamma^0\gamma^k\gamma^0\gamma^5 u = +\overline{u}\gamma^k\gamma^5 u = +j_A^k.$$

Therefore, the scalar product of two axial vector currents is invariant under parity transformations

$$j_1 \cdot j_2 = j_1^0 j_2^0 - j_1^k j_2^k \xrightarrow{\hat{P}} (-j_1^0)(-j_2^0) - j_1^k j_2^k = j_1 \cdot j_2. \tag{11.7}$$

This should come as no surprise, the matrix element is a scalar quantity; if it is formed from the four-vector scalar product of either two vectors or two axial vectors it has to be invariant under the parity transformation.

To summarise, the parity transformation properties of the components of the vector and the axial vector currents are

$$j_V^0 \xrightarrow{\hat{P}} +j_V^0, \quad j_V^k \xrightarrow{\hat{P}} -j_V^k, \quad \text{and} \quad j_A^0 \xrightarrow{\hat{P}} -j_A^0, \quad j_A^k \xrightarrow{\hat{P}} +j_A^k.$$

Whilst the scalar products of two vector currents or two axial vector currents are unchanged in a parity transformation, the scalar product $j_V \cdot j_A$ transforms to $-j_V \cdot j_A$. Hence the combination of vector *and* axial vector currents provides a mechanism to explain the observed parity violation in the weak interaction.

Consider the (inverse-β-decay) charged-current weak interaction process $\nu_e d \to e^- u$, shown in Figure 11.3, with assumed currents of the form

$$j_{\nu e}^\mu = \overline{u}(p_3)(g_V\gamma^\mu + g_A\gamma^\mu\gamma^5)u(p_1) = g_V j_{\nu e}^V + g_A j_{\nu e}^A,$$
$$j_{du}^\nu = \overline{u}(p_4)(g_V\gamma^\nu + g_A\gamma^\nu\gamma^5)u(p_2) = g_V j_{du}^V + g_A j_{du}^A.$$

The matrix element is proportional to the four-vector scalar products of two currents

$$\mathcal{M}_{fi} \propto j_{\nu e} \cdot j_{du} = g_V^2 j_{\nu e}^V \cdot j_{du}^V + g_A^2 j_{\nu e}^A \cdot j_{du}^A + g_V g_A (j_{\nu e}^V \cdot j_{du}^A + j_{\nu e}^A \cdot j_{du}^V).$$

The terms $j_{\nu e}^V \cdot j_{\nu e}^V$ and $j_{\nu e}^A \cdot j_{\nu e}^A$ do not change sign under a parity transformation, but the mixed V and A combinations do, and therefore

$$j_{\nu e} \cdot j_{du} \xrightarrow{\hat{P}} g_V^2 j_{\nu e}^V \cdot j_{du}^V + g_A^2 j_{\nu e}^A \cdot j_{du}^A - g_V g_A (j_{\nu e}^V \cdot j_{du}^A + j_{\nu e}^A \cdot j_{du}^V).$$

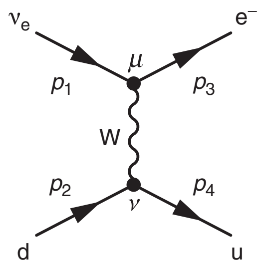

Fig. 11.3 The lowest-order Feynman diagram for the charged-current weak interaction $\nu_e d \to e^- u$.

Thus the relative strength of the parity violating part of the matrix element compared to the parity conserving part is given by

$$\frac{g_V g_A}{g_V^2 + g_A^2}.$$

Hence, if either g_V or g_A is zero, parity is conserved in the interaction. Furthermore, maximal parity violation occurs when $|g_V| = |g_A|$, corresponding to a pure $V - A$ or $V + A$ interaction. From experiment, it is known that the weak charged current due to the exchange of $W^\pm$ bosons is a vector minus axial vector $(V - A)$ interaction of the form $\gamma^\mu - \gamma^\mu\gamma^5$, with a vertex factor of

$$\frac{-ig_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5). \tag{11.8}$$

Here g_W is the weak coupling constant (which is often written simply as g). The origin of the additional numerical factors will be explained in Chapter 15. The corresponding four-vector current is given by

$$j^\mu = \frac{g_W}{\sqrt{2}}\overline{u}(p')\frac{1}{2}\gamma^\mu(1-\gamma^5)u(p).$$

11.4 Chiral structure of the weak interaction

In Chapter 6, the left- and right-handed *chiral* projection operators,

$$P_R = \frac{1}{2}(1+\gamma^5) \quad \text{and} \quad P_L = \frac{1}{2}(1-\gamma^5),$$

were introduced. Any spinor can be decomposed into left- and right-handed chiral components,

$$u = \frac{1}{2}(1+\gamma^5)u + \frac{1}{2}(1-\gamma^5)u = P_R u + P_L u = a_R u_R + a_L u_L,$$

with coefficients a_R and a_L. In Section 6.4.1, it was shown that only two combinations of chiral spinors (RR and LL) gave non-zero values for the QED *vector* current, $\overline{u}(p')\gamma^{\mu}u(p)$. For the weak interaction, the $V - A$ vertex factor of (11.8) already includes the left-handed chiral projection operator,

$$\tfrac{1}{2}(1 - \gamma^5).$$

In this case, the current where both the spinors are right-handed chiral states is also zero

$$\begin{aligned} j^{\mu}_{RR} &= \frac{g_W}{\sqrt{2}}\,\overline{u}_R(p')\gamma^{\mu}\tfrac{1}{2}(1-\gamma^5)u_R(p) \\ &= \frac{g_W}{\sqrt{2}}\,\overline{u}_R(p')\gamma^{\mu}P_L u_R(p) = 0, \end{aligned}$$

and the only non-zero current for particle spinors involves only left-handed chiral states. Hence only left-handed chiral *particle* states participate in the charged-current weak interaction. For antiparticle spinors P_L projects out right-handed chiral states,

$$\tfrac{1}{2}(1-\gamma^5)v = v_R,$$

and therefore only right-handed chiral *antiparticle* states participate in the charged-current weak interaction. In the limit $E \gg m$, where the chiral and helicity states are the same, the $V-A$ term in the weak interaction vertex projects out left-handed helicity particle states and right-handed helicity antiparticles states. Hence, in this ultra-relativistic limit, the only allowed helicity combinations for the weak interaction vertices involving electrons/positrons and electron neutrinos/antineutrinos are those shown in Figure 11.4.

The maximally different coupling of the weak charged-current interaction to left-handed and right-handed chiral states is the origin of parity violation. For example, the left-hand plot of Figure 11.5 shows the helicity configuration of the allowed weak interaction of a high-energy left-handed e^- and a right-handed $\overline{\nu}_e$. In the parity mirror, the vector quantities are reversed, $\mathbf{p} \rightarrow -\mathbf{p}$, but the axial vector spins of the particles remain unchanged, giving a RH particle and a LH antiparticle. Hence the parity operation transforms an allowed weak interaction into one that is not allowed, maximally violating the conservation of parity.

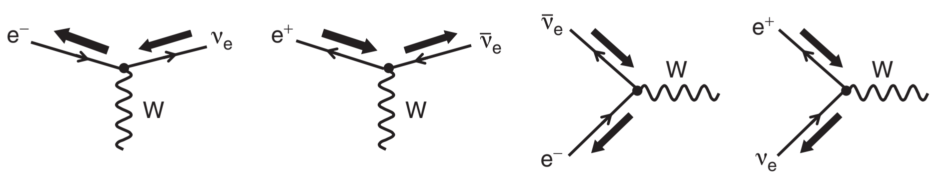

Fig. 11.4 The allowed helicity combinations in weak interaction vertices involving the e^+, e^-, ν_e and $\overline{\nu}_e$, in the limit where $E \gg m$ (where the helicity states are effectively the same as the chiral states).

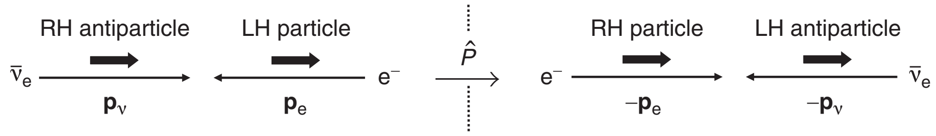

Fig. 11.5 The allowed helicity combination for a $e^-\nu_e$ weak interaction, and (right) its parity transformed equivalent.

11.5 The W-boson propagator

The Feynman rule for the propagator of QED, corresponding to the exchange of the massless spin-1 photon, is

$$\frac{-ig_{\mu\nu}}{q^2}.$$

The weak interaction not only differs from QED and QCD in the form of the interaction vertex, but it is mediated by the massive W bosons, with $m_W \sim 80\,\text{GeV}$. Consequently, the q^2-dependence of the W-boson propagator is given by (5.7),

$$\frac{1}{q^2 - m_W^2}.$$

The $g_{\mu\nu}$ term in the Feynman rule for QED propagator is associated with the sum over the polarisation states of the virtual photon,

$$\sum_\lambda \epsilon_\mu^{\lambda *}\epsilon_\nu^\lambda = -g_{\mu\nu}.$$

Massive spin-1 particles differ from massless spin-1 particles in having the additional degree of freedom of a longitudinal polarisation state. In Appendix D, it is shown that the corresponding sum over the polarisation states of the exchanged virtual massive spin-1 boson gives

$$\sum_\lambda \epsilon_\mu^{\lambda *}\epsilon_\nu^\lambda = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_W^2}.$$

Therefore, the Feynman rule associated with the exchange of a virtual W boson is

$$\frac{-i}{q^2 - m_W^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2}\right). \tag{11.9}$$

In the limit where $q^2 \ll m_W^2$, the $q_\mu q_\nu$ term is small and the propagator can be taken to be

$$\frac{-ig_{\mu\nu}}{q^2 - m_W^2}. \tag{11.10}$$

More generally, for the lowest-order calculations in the following chapters the $q_\mu q_\nu$ term in (11.9) does not contribute to the matrix element squared and it is sufficient to take the propagator term to be that given in (11.10).

11.5.1 Fermi theory

For most low-energy weak interactions, such as the majority of particle decays, $|q^2| \ll m_W^2$ and the W-boson propagator of (11.10) can be approximated by

$$i\frac{g_{\mu\nu}}{m_W^2}, \tag{11.11}$$

and the effective interaction no longer has any q^2 dependence. Physically this corresponds to replacing the propagator with an interaction which occurs at a single point in space-time, as indicated in Figure 11.6. Hence, in the low-energy limit, the weak charged-current can be expressed in terms of this four-fermion contact interaction.

The original description of the weak interaction, due to Fermi (1934), was formulated before the discovery of the parity violation and the matrix element for β-decay was expressed in terms of a contact interaction

$$\mathcal{M}_{fi} = G_F\, g_{\mu\nu}[\overline{\psi}_3\gamma^\mu\psi_1][\overline{\psi}_4\gamma^\nu\psi_2], \tag{11.12}$$

where the strength of the weak interaction is given by the Fermi constant G_F. After the discovery of parity violation by Wu *et al.* (1957), this expression was modified to

$$\mathcal{M}_{fi} = \frac{1}{\sqrt{2}}G_F\, g_{\mu\nu}[\overline{\psi}_3\gamma^\mu(1-\gamma^5)\psi_1][\overline{\psi}_4\gamma^\nu(1-\gamma^5)\psi_2], \tag{11.13}$$

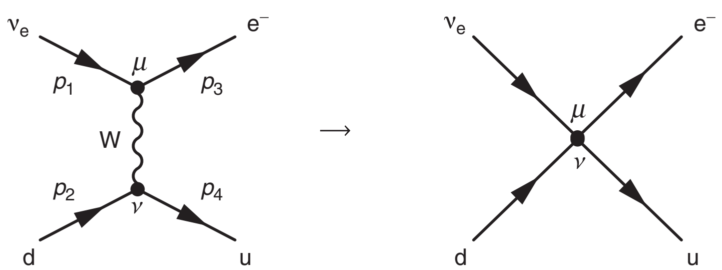

Fig. 11.6 The weak interaction Feynman diagram and the $q^2 \ll m_W^2$ limit of an effective contact interaction.

where the factor of $\sqrt{2}$ was introduced so that the numerical value of G_F did not need to be changed. The expression of (11.13) can be compared to the full expression obtained using the Feynman rules for the weak interaction,

$$\mathcal{M}_{fi} = -\left[\frac{g_W}{\sqrt{2}}\overline{\psi}_3\frac{1}{2}\gamma^\mu(1-\gamma^5)\psi_1\right]\cdot\left[\frac{g_{\mu\nu}-q_\mu q_\nu/m_W^2}{q^2-m_W^2}\right]\cdot\left[\frac{g_W}{\sqrt{2}}\overline{\psi}_4\frac{1}{2}\gamma^\nu(1-\gamma^5)\psi_2\right],$$

which in the limit of $q^2 \ll m_W^2$ reduces to

$$M_{fi} = \frac{g_W^2}{8m_W^2}g_{\mu\nu}[\overline{\psi}_3\gamma^\mu(1-\gamma^5)\psi_1][\overline{\psi}_4\gamma^\nu(1-\gamma^5)\psi_2]. \tag{11.14}$$

Hence, by comparing (11.13) and (11.14), it can be seen that the Feynman rules in the low-q^2 limit, give the same expression for the matrix element as obtained from Fermi theory and therefore the Fermi constant is related to the weak coupling strength by

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}. \tag{11.15}$$

Strength of the weak interaction

The strength of the weak interaction is most precisely determined from low-energy measurements, and in particular from the muon lifetime. For these low-energy measurements, where for example $m_\mu \ll m_W$, Fermi theory can be used. The calculation of the decay rate for $\mu^- \to e^-\nu_\mu\overline{\nu}_e$ includes a fairly involved integration over the three-body phase space of the final state and the results are simply quoted here. The muon lifetime τ_μ is related to its mass by

$$\Gamma(\mu^- \to e^-\nu_\mu\overline{\nu}_e) = \frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}. \tag{11.16}$$

The precise measurements of the muon lifetime and mass,

$$m_\mu = 0.105\,658\,371\,5(35)\,\text{GeV} \quad \text{and} \quad \tau_\mu = 2.196\,981\,1(22)\times 10^{-6}\,\text{s},$$

provide a precise determination of the Fermi constant,

$$G_F = 1.166\,38\times 10^{-5}\,\text{GeV}^{-2}.$$

However, G_F does not express the fundamental strength of the weak interaction, it is related to the ratio of the coupling strength g_W and the W-boson mass by (11.15). Nevertheless, G_F is the quantity that is precisely measured in muon decay and it is still used parameterise the strength of weak interaction.

The value of fundamental coupling constant g_W can be obtained from G_F using the precise measurement of $m_W = 80.385 \pm 0.015$ GeV (see Chapter 16). From the relation of (11.15) and the measured values of G_F and m_W, the dimensionless coupling constant of the weak interaction is

$$\alpha_W = \frac{g_W^2}{4\pi} = \frac{8m_W^2 G_F}{4\sqrt{2}\pi} \approx \frac{1}{30}.$$

Hence the weak interaction is in fact intrinsically stronger than the electromagnetic interaction, $\alpha_W > \alpha$. It is only the presence of the large mass of the W boson in the propagator that is responsible for the weakness of the low-energy weak interaction compared to that of QED. For a process where the exchanged boson carries four-momentum q, where $|q^2| \ll m_W^2$, the QED and weak interactions propagators are respectively

$$P_{QED} \sim \frac{1}{q^2} \quad \text{and} \quad P_W \sim \frac{1}{q^2 - m_W^2} \approx -\frac{1}{m_W^2}.$$

Therefore weak interaction decay rates, which are proportional to $|\mathcal{M}|^2$, are suppressed by a factor q^4/m_W^4 relative to QED decay rates. In contrast, in the high-energy limit where $|q^2| > m_W^2$, the m_W^2 term in the weak propagator is relatively unimportant and the electromagnetic and weak interactions have similar strength, as will be seen directly in the results from high-Q^2 electron–proton interactions, described in Section 12.5.

11.6 Helicity in pion decay

The charged pions ($\pi^\pm$) are the $J^P = 0^-$ meson states formed from $u\overline{d}$ and $d\overline{u}$. They are the lightest mesons with $m(\pi^\pm) \sim 140\,\text{MeV}$ and therefore cannot decay via the strong interaction; they can only decay through the weak interaction to final states with lighter fundamental fermions. Hence charged pions can only decay to final states with either electrons or muons. The three main decay modes of the π^- are the charged-current weak processes $\pi^- \to e^-\overline{\nu}_e$, $\pi^- \to \mu^-\overline{\nu}_\mu$ and $\pi^- \to \mu^-\overline{\nu}_\mu\gamma$, with decays to $\mu^-\overline{\nu}_\mu$ dominating.

The Feynman diagrams for the decays $\pi^- \to e^-\overline{\nu}_e$ and $\pi^- \to \mu^-\overline{\nu}_\mu$ are shown in Figure 11.7. Because the strength of the weak interaction for the different lepton generations is found to be the same (see Chapter 12), it might be expected that the matrix elements for the decays $\pi^- \to e^-\overline{\nu}_e$ and $\pi^- \to \mu^-\overline{\nu}_\mu$ would be similar. For a two-body decay, the phase space factor is proportional the momentum of the decay products in the centre-of-mass frame, see (3.49). On this basis, the decay rate to $e^-\overline{\nu}_e$ would be expected to be greater than that to $\mu^-\overline{\nu}_\mu$. However, the opposite is found to be true; charged pions decay almost entirely by $\pi^- \to \mu^-\overline{\nu}_\mu$

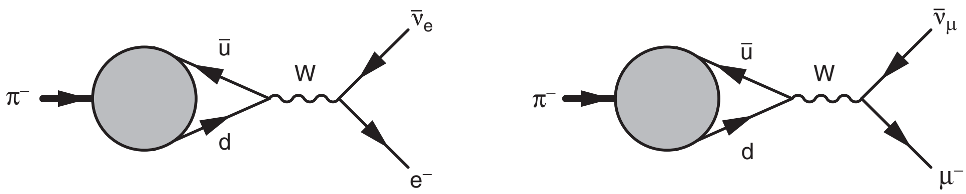

Fig. 11.7 Two of the three main decay modes for the π^-. The decay $\pi^- \to \mu^-\overline{\nu}_\mu\gamma$ (not shown) has a comparable branching ratio to that for $\pi^- \to e^-\overline{\nu}_e$.

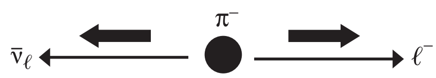

Fig. 11.8 The helicity configuration in $\pi^- \to \ell^-\overline{\nu}_\ell$ decay, where $\ell = e$ or μ.

(or equivalently $\pi^+ \to \mu^+\nu_\mu$) with a branching ratio of 99.988% and the measured ratio of the decay rates to electrons and muons is

$$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = 1.230(4) \times 10^{-4}.$$

This counterintuitive result is a manifestation of the chiral structure of the weak interaction and provides a clear illustration of the difference between helicity, defined by $\boldsymbol{\sigma} \cdot \mathbf{p}/|\mathbf{p}|$, and chirality defined by the action of the chiral projection operators.

The weak interaction only couples to LH chiral particle states and RH chiral antiparticle states. Because neutrinos are effectively massless, $m_\nu \ll E$, the neutrino chiral states are, in all practical circumstances, equivalent to the helicity states. Therefore, the antineutrino from a π^- decay is always produced in a RH helicity state. Because the pion is a spin-0 particle, the lepton–neutrino system must be produced in the spin-0 singlet state, with the charged lepton and neutrino spins in opposite directions. Therefore, because the neutrino is RH, conservation of angular momentum implies that the charged lepton is also produced in a RH helicity state, and the only allowed spin configuration is that of Figure 11.8. Since the weak interaction vertex is non-zero only for LH chiral particle states, the charged lepton has, in some sense, the "wrong helicity" for the weak interaction. If the charged leptons were also massless, the decay would not occur. However, chiral and helicity states are not equivalent and the weak decay to a RH *helicity* particle state can occur, although it may be highly suppressed.

In general the RH helicity spinor $u_\uparrow$ can be decomposed into RH and LH chiral components, u_R and u_L, given by (6.38),

$$u_\uparrow \equiv \tfrac{1}{2}\left(1 + \frac{p}{E+m}\right)u_R + \tfrac{1}{2}\left(1 - \frac{p}{E+m}\right)u_L. \tag{11.17}$$

In the weak interaction vertex only the u_L component of (11.17) will give a non-zero contribution to the matrix element. Putting aside the relatively small differences from the normalisations of the lepton and neutrino spinors, the charged-current weak decay matrix element is proportional to the size of the LH chiral component in (11.17) and

$$\mathcal{M} \sim \frac{1}{2}\left(1 - \frac{\mathrm{p}_\ell}{E_\ell + m_\ell}\right), \tag{11.18}$$

where E_ℓ, p_ℓ and m_ℓ are the energy, momentum and mass of the charged lepton. If the charged lepton is highly relativistic, the left-handed chiral component of the right-handed helicity state will be very small, resulting in a suppression of the decay rate.

Taking the mass of the neutrino to be zero, it is straightforward to show that

$$E_\ell = \frac{m_\pi^2 + m_\ell^2}{2m_\pi} \quad \text{and} \quad \mathrm{p}_\ell = \frac{m_\pi^2 - m_\ell^2}{2m_\pi}, \tag{11.19}$$

giving

$$\frac{\mathrm{p}_\ell}{E_\ell + m_\ell} = \frac{m_\pi - m_\ell}{m_\pi + m_\ell},$$

which when substituted into (11.18), demonstrates that

$$\mathcal{M} \sim \frac{m_\ell}{m_\pi + m_\ell}.$$

Because $m_\mu/m_e \approx 200$, pion decays to electrons are strongly suppressed with respect to those to muons. This helicity suppression reflects the fact that the electrons produced in pion decay are highly relativistic, $\beta = 0.999\,97$, and therefore the chiral states almost correspond to the helicity states. For the decay to muons, $\beta = 0.27$, and the u_L coefficient in (11.17) is significant. The above discussion gives a qualitative explanation of why charged pions predominantly decay to muons rather than electrons. The full calculation, which is interesting in its own right, is given below.

11.6.1 Pion decay rate

Consider the $\pi^- \to \ell^-\overline{\nu}_\ell$ decay in its rest frame, where the direction of the charged lepton defines the z-axis, as shown in Figure 11.9. In this case, the four-momenta of the π^-, ℓ^- and $\overline{\nu}_\ell$ are respectively,

$$p_\pi = (m_\pi, 0, 0, 0), \quad p_\ell = p_3 = (E_\ell, 0, 0, \mathrm{p}) \quad \text{and} \quad p_{\overline{\nu}} = p_4 = (\mathrm{p}, 0, 0, -\mathrm{p}),$$

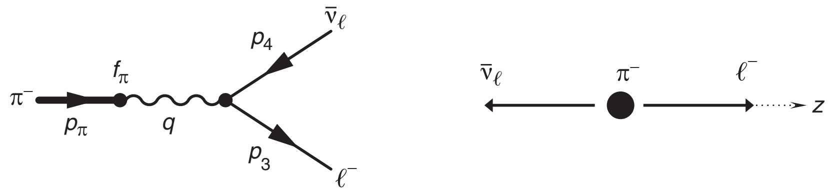

Fig. 11.9 The definition of the four-momenta in the process $\pi^- \to \ell^-\overline{\nu}_\ell$.

where p is the magnitude of the momentum of both the charged lepton and antineutrino in the centre-of-mass frame.

The weak leptonic current associated with the $\ell^-\overline{\nu}_\ell$ vertex is

$$j_\ell^\nu = \frac{g_W}{\sqrt{2}}\overline{u}(p_3)\tfrac{1}{2}\gamma^\nu(1-\gamma^5)v(p_4).$$

Because the pion is a bound $q\overline{q}$ state, the corresponding hadronic current cannot be expressed in terms of free particle Dirac spinors. However, the pion current has to be a four-vector such that the four-vector scalar product with the leptonic current gives a Lorentz-invariant expression for the matrix element. Since the pion is a spin-0 particle, the only four-vector quantity that can be used is the pion four-momentum. Hence, the most general expression for the pion current is obtained by replacing $\overline{v}\gamma^\mu(1-\gamma^5)u$ with $f_\pi p_\pi^\mu$, where f_π is a constant associated with the decay. The matrix element for the decay $\pi^- \to \ell^-\overline{\nu}_\ell$ therefore can be written as

$$\begin{aligned}\mathcal{M}_{fi} &= \left[\frac{g_W}{\sqrt{2}}\frac{1}{2}f_\pi p_\pi^\mu\right]\times\left[\frac{g_{\mu\nu}}{m_W^2}\right]\times\left[\frac{g_W}{\sqrt{2}}\overline{u}(p_3)\gamma^\nu\tfrac{1}{2}(1-\gamma^5)v(p_4)\right]\\ &= \frac{g_W^2}{4m_W^2}g_{\mu\nu}f_\pi p_\pi^\mu\overline{u}(p_3)\gamma^\nu\tfrac{1}{2}(1-\gamma^5)v(p_4),\end{aligned}$$

where the propagator has been approximated by the Fermi contact interaction (which is an extremely good approximation because $q^2 = m_\pi^2 \ll m_W^2$). In the pion rest frame, only the time-like component of the pion four-momentum is non-zero, $p_\pi^0 = m_\pi$, and hence

$$\mathcal{M}_{fi} = \frac{g_W^2}{4m_W^2}f_\pi m_\pi\overline{u}(p_3)\gamma^0\tfrac{1}{2}(1-\gamma^5)v(p_4).$$

Because $\overline{u}\gamma^0 = u^\dagger\gamma^0\gamma^0 = u^\dagger$, this can be written as

$$\mathcal{M}_{fi} = \frac{g_W^2}{4m_W^2}f_\pi m_\pi u^\dagger(p_3)\tfrac{1}{2}(1-\gamma^5)v(p_4). \tag{11.20}$$

For the neutrino, which has $m \ll E$, the helicity eigenstates are essentially equivalent to the chiral states and therefore $\frac{1}{2}(1-\gamma^5)v(p_4) = v_\uparrow(p_4)$, and thus (11.20) becomes

$$\mathcal{M}_{fi} = \frac{g_W^2}{4m_W^2} f_\pi m_\pi u^\dagger(p_3) v_\uparrow(p_4). \tag{11.21}$$

The spinors corresponding to the two possible helicity states of the charged lepton spinor are obtained from (4.65) with $(\theta = 0, \phi = 0)$,

$$u_\uparrow(p_3) = \sqrt{E_\ell + m_\ell}\begin{pmatrix} 1 \\ 0 \\ \frac{\mathrm{p}}{E_\ell + m_\ell} \\ 0 \end{pmatrix} \quad \text{and} \quad u_\downarrow(p_3) = \sqrt{E_\ell + m_\ell}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{\mathrm{p}}{E_\ell + m_\ell} \end{pmatrix}, \tag{11.22}$$

and the right-handed antineutrino spinor is given by (4.66) with $(\theta = \pi, \phi = \pi)$,

$$v_\uparrow(p_4) = \sqrt{\mathrm{p}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{11.23}$$

From (11.22) and (11.23) it is immediately clear that $u_\downarrow^\dagger(p_3)v_\uparrow(p_4) = 0$. Therefore, as anticipated, of the four possible helicity combinations, the only non-zero matrix element corresponds to the case where both the charged lepton and the antineutrino are in RH helicity states. Using the explicit forms for the spinors, the matrix element of (11.21) is

$$\mathcal{M}_{fi} = \frac{g_W^2}{4m_W^2} f_\pi m_\pi \sqrt{E_\ell + m_\ell}\,\sqrt{\mathrm{p}}\left(1 - \frac{\mathrm{p}}{E_\ell + m_\ell}\right). \tag{11.24}$$

Equation (11.24) can be simplified using the expressions for E_ℓ and p given in (11.19), such that

$$\begin{aligned}\mathcal{M}_{fi} &= \frac{g_W^2}{4m_W^2} f_\pi m_\pi \cdot \frac{m_\pi + m_\ell}{\sqrt{2m_\pi}} \cdot \left(\frac{m_\pi^2 - m_\ell^2}{2m_\pi}\right)^{\frac{1}{2}} \cdot \frac{2m_\ell}{m_\pi + m_\ell} \\ &= \left(\frac{g_W}{2m_W}\right)^2 f_\pi m_\ell (m_\pi^2 - m_\ell^2)^{\frac{1}{2}}.\end{aligned}$$

Since the pion is a spin-0 particle, there is no need to average over the initial-state spins, and the matrix element squared is given by

$$\langle|\mathcal{M}_{fi}|^2\rangle \equiv |\mathcal{M}_{fi}|^2 = 2G_F^2 f_\pi^2 m_\ell^2(m_\pi^2 - m_\ell^2),$$

where g_W has been expressed in terms of G_F using (11.15). Finally, the decay rate can be determined from the expression for the two-body decay rate given by (3.49), where the integral over solid angle introduces a factor of 4π as there is no angular dependence in $\langle|\mathcal{M}_{fi}|^2\rangle$. Hence

$$\Gamma = \frac{4\pi}{32\pi^2 m_\pi^2}\,\mathrm{p}\,\langle|\mathcal{M}_{fi}|^2\rangle = \frac{G_F^2}{8\pi m_\pi^3} f_\pi^2\left[m_\ell(m_\pi^2 - m_\ell^2)\right]^2, \qquad (11.25)$$

where p is given by (11.19). Therefore, to lowest order, the predicted ratio of the $\pi^- \to e^-\overline{\nu}_e$ to $\pi^- \to \mu^-\overline{\nu}_\mu$ decay rates is

$$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = \left[\frac{m_e(m_\pi^2 - m_e^2)}{m_\mu(m_\pi^2 - m_\mu^2)}\right]^2 = 1.26\times10^{-4},$$

which is in reasonable agreement with the measured value of $1.230(4)\times10^{-4}$.

11.7 Experimental evidence for $V-A$

The $V-A$ nature of the weak interaction is an experimentally established fact. For example, if the weak interaction was a scalar ($\overline{\psi}\phi$) or pseudoscalar ($\overline{\psi}\gamma^5\phi$) interaction, the predicted ratio of the charged pion leptonic decay rates would be $\Gamma(\pi^- \to e^-\overline{\nu}_e)/\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu) = 5.5$, in clear contradiction with the experimental observations. In general, any weak decay can be expressed in terms of a linear combination of the five bilinear covariants, scalar (S), pseudoscalar (P), vector (V), axial vector (A) and tensor (T):

$$g_S\overline{\psi}\phi,\;\; g_P\overline{\psi}\gamma^5\phi,\;\; g_V\overline{\psi}\gamma^\mu\phi,\;\; g_A\overline{\psi}\gamma^\mu\gamma^5\phi \;\text{ and }\; g_T\overline{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\phi.$$

By comparing these predictions with the experimental measurements, limits can be placed on the possible sizes of the different contributions. The most precise test of the $V-A$ structure of the weak interaction is based on measurements of the angular distribution of decays of approximately 10^{10} polarised muons by the TWIST experiment: see Bayes *et al.* (2011). The measurements are expressed in terms of the Michel parameters which parameterise the general combination of the possible $S+P+V+A+T$ interaction terms. For example, the Michel parameter ρ, which for a pure $V-A$ interaction should be 0.75, is measured to be $\rho = 0.749\,97 \pm 0.000\,26$. All such tests indicate that the charged-current weak interaction is described by a $V-A$ vertex factor.

Summary

In this chapter, the general structure of the weak charged-current interaction was introduced. Unlike QED and QCD, the weak interaction does not conserve parity. Parity violation in the weak charged-current interaction is a direct consequence of the $V - A$ form of the weak charged-current, which treats left-handed and right-handed particles differently. The weak charged-current vertex factor was found to be

$$\frac{-ig_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5),$$

and the propagator associated with the exchange of the massive W bosons is

$$\frac{-i}{q^2 - m_W^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right).$$

Because of the $V - A$ interaction only

LH chiral particle states and RH chiral antiparticle states

participate in the weak charged-current.

Problems

11.1 Explain why the strong decay $\rho^0 \to \pi^-\pi^+$ is observed, but the strong decay $\rho^0 \to \pi^0\pi^0$ is not.

Hint: you will need to consider conservation of angular momentum, parity and the symmetry of the $\pi^0\pi^0$ wavefunction.

11.2 When π^- mesons are stopped in a deuterium target they can form a bound $(\pi^- - \mathrm{D})$ state with zero orbital angular momentum, $\ell = 0$. The bound state decays by the strong interaction

$$\pi^-\mathrm{D} \to \mathrm{nn}.$$

By considering the possible spin and orbital angular momentum states of the nn system, and the required symmetry of the wavefunction, show that the pion has negative intrinsic parity.

Note: the deuteron has $J^P = 1^+$ and the pion is a spin-0 particle.

11.3 Classify the following quantities as either scalars (S), pseudoscalars (P), vectors (V) or axial-vectors (A):

(a) mechanical power, $P = \mathbf{F}\cdot\mathbf{v}$;
(b) force, $\mathbf{F}$;
(c) torque, $\mathbf{G} = \mathbf{r}\times\mathbf{F}$;
(d) vorticity, $\mathbf{\Omega} = \nabla\times\mathbf{v}$;

(e) magnetic flux, $\phi = \int \mathbf{B} \cdot d\mathbf{S}$;
(f) divergence of the electric field strength, $\nabla \cdot \mathbf{E}$.

11.4 In the annihilation process $e^+e^- \to q\overline{q}$, the QED vector interaction leads to non-zero matrix elements only for the chiral combinations $LR \to LR$, $LR \to RL$, $RL \to RL$, $RL \to LR$. What are the corresponding allowed chiral combinations for S, P and $S - P$ interactions?

11.5 Consider the decay at rest $\tau^- \to \pi^-\nu_\tau$, where the spin of the tau is in the positive z-direction and the ν_τ and π^- travel in the $\pm z$-directions. Sketch the allowed spin configurations assuming that the form of the weak charged-current interaction is (i) $V - A$ and (ii) $V + A$.

11.6 Repeat the pion decay calculation for a pure scalar interaction and show that the predicted ratio of decay rates is

$$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} \approx 5.5.$$

11.7 Predict the ratio of the $K^- \to e^-\overline{\nu}_e$ and $K^- \to \mu^-\overline{\nu}_\mu$ weak interaction decay rates and compare your answer to the measured value of

$$\frac{\Gamma(K^- \to e^-\overline{\nu}_e)}{\Gamma(K^- \to \mu^-\overline{\nu}_\mu)} = (2.488 \pm 0.012) \times 10^{-5}.$$

11.8 Charged kaons have several weak interaction decay modes, the largest of which are

$$K^+(u\overline{s}) \to \mu^+\nu_\mu, \quad K^+ \to \pi^+\pi^0 \quad \text{and} \quad K^+ \to \pi^+\pi^+\pi^-.$$

(a) Draw the Feynman diagrams for these three weak decays.
(b) Using the measured branching ratio

$$Br(K^+ \to \mu^+\nu_\mu) = 63.55 \pm 0.11\,\%,$$

estimate the lifetime of the charged kaon.

Note: charged pions decay almost 100% of the time by the weak interaction $\pi^+ \to \mu^+\nu_\mu$ and have a lifetime of $(2.6033 \pm 0.0005) \times 10^{-8}$ s.

11.9 From the prediction of (11.25) and the above measured value of the charged pion lifetime, obtain a value for f_π.

11.10 Calculate the partial decay width for the decay $\tau^- \to \pi^-\nu_\tau$ in the following steps.

(a) Draw the Feynman diagram and show that the corresponding matrix element is

$$\mathcal{M} \approx \sqrt{2}G_F f_\pi \overline{u}(p_\nu)\gamma^\mu \tfrac{1}{2}(1 - \gamma^5)u(p_\tau)g_{\mu\nu}p_\pi^\nu.$$

(b) Taking the τ^- spin to be in the z-direction and the four-momentum of the neutrino to be

$$p_\nu = p^*(1, \sin\theta, 0, \cos\theta),$$

show that the leptonic current is

$$j^\mu = \sqrt{2m_\tau p^*}\,(-s, -c, -ic, s)\,,$$

where $s = \sin\left(\frac{\theta}{2}\right)$ and $c = \cos\left(\frac{\theta}{2}\right)$. Note that, for this configuration, the spinor for the τ^- can be taken to be u_1 for a particle at rest.

(c) Write down the four-momentum of the π^- and show that

$$|\mathcal{M}|^2 = 4G_F^2 f_\pi^2 m_\tau^3 p^* \sin^2\left(\frac{\theta}{2}\right).$$

(d) Hence show that

$$\Gamma(\tau^- \to \pi^- \nu_\tau) = \frac{G_F^2 f_\pi^2}{16\pi} m_\tau^3 \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau^2}\right)^2.$$

(e) Using the value of f_π obtained in the previous problem, find a numerical value for $\Gamma(\tau^- \to \pi^- \nu_\tau)$.

(f) Given that the lifetime of the τ-lepton is measured to be $\tau_\tau = 2.906 \times 10^{-13}$ s, find an approximate value for the $\tau^- \to \pi^- \nu_\tau$ branching ratio.