Determination of the CKM Matrix elements

- In total, there are 9 matrix elements to determine:

$$V_{\mathrm{CKM}} \equiv V_L^u V_L^{d\dagger} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

CKM Matrix parameterization

- The CKM matrix is unitary: can be parameterized by 3 mixing angles and the CP-violating phase
- A standard choice is the same as was shown for the PMNS matrix:

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

- Since experimentally $s_{13} \ll s_{23} \ll s_{12} \ll 1$, the matrix hierarchy is exhibited using the Wolfenstein parameterization (written in terms of λ, A, ρ, η):

$\lambda = S_{12}$ $\quad C_{12} = \sqrt{1 - S_{12}^2} = \sqrt{1-\lambda^2} = 1 - \frac{\lambda^2}{2}$

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

$(1+x)^\alpha \approx 1 + \alpha x$

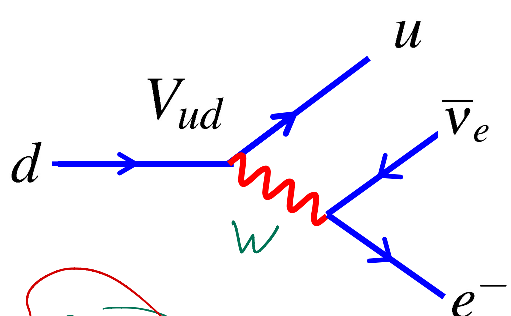
$x \ll 1$

Determination of the CKM Matrix elements

- The experimental determination of the CKM matrix elements comes mainly from measurements of leptonic decays (the leptonic part is well understood).
- It is easy to produce/observe meson decays, however theoretical uncertainties associated with the decays of bound states often limits the precision
- Contrast this with the measurements of the PMNS matrix, where there are few theoretical uncertainties and the experimental difficulties in dealing with neutrinos limits the precision.

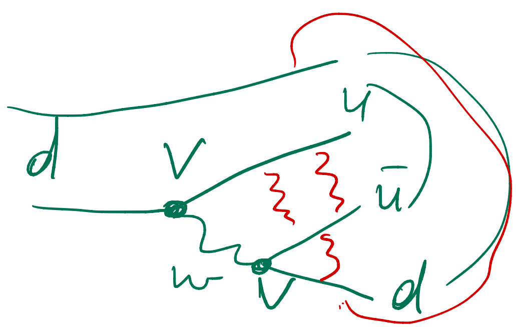
1 $\boxed{|V_{ud}|}$ from nuclear beta decay

$$\begin{pmatrix} \times & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$



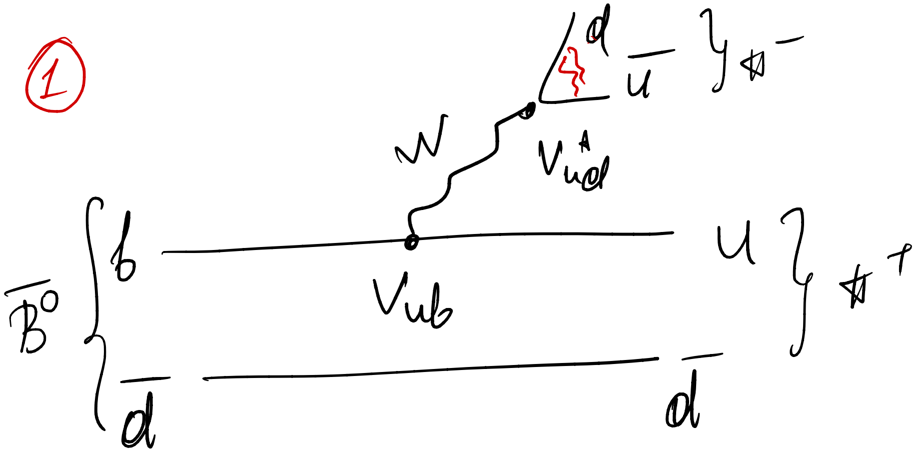
Super-allowed $0^+ \rightarrow 0^+$ beta decays are relatively free from theoretical uncertainties

$$\Gamma \propto |V_{ud}|^2$$



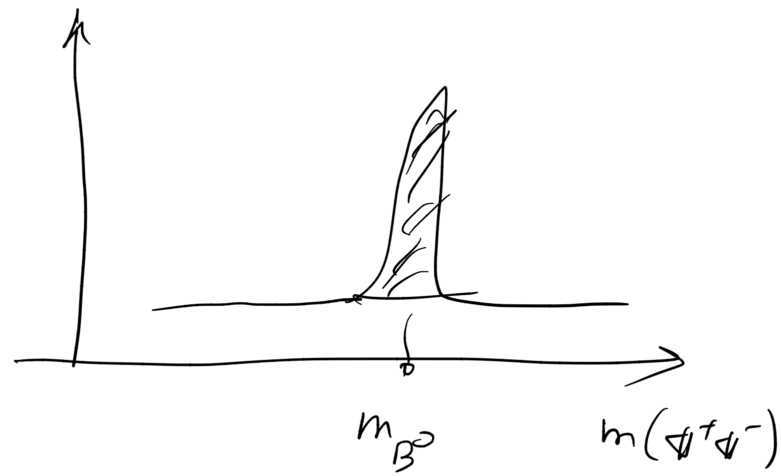
$$\boxed{|V_{ud}| = 0.97370 \pm 0.00014} \qquad (\approx \cos\theta_c)$$

1

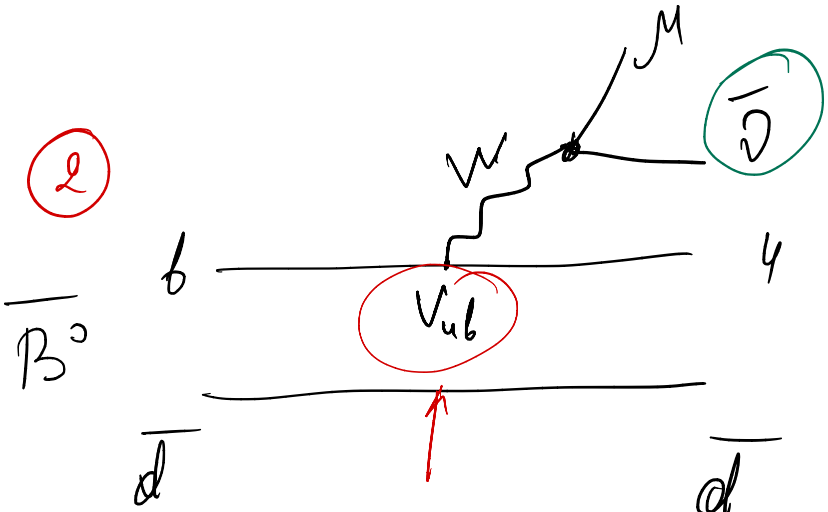


$\bar{B}^0 \to \pi^+\pi^-$

→ easy exp.



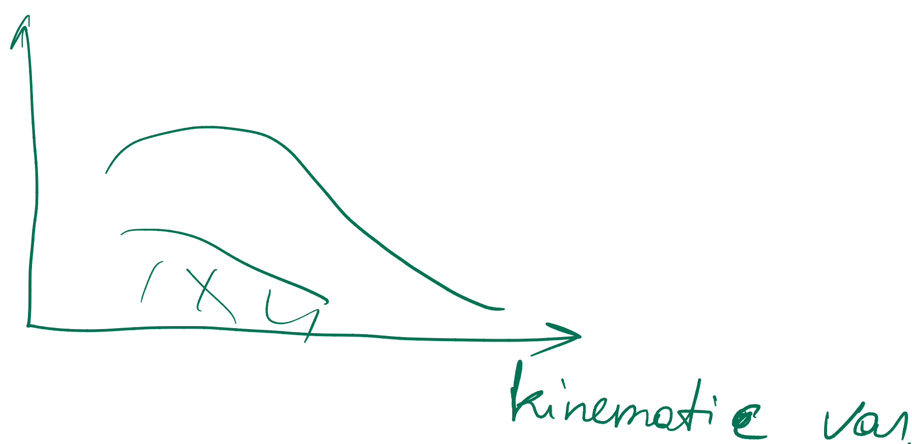
2



– easy theoret.

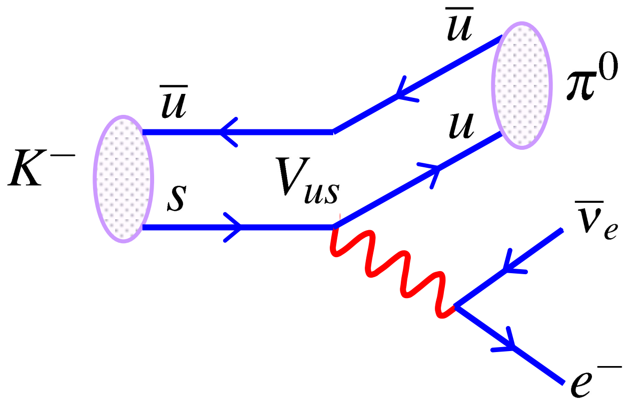
small

$1^{st} \leftrightarrow 3^{rd}$



2 $|V_{us}|$ from semi-leptonic kaon decays

$$\begin{pmatrix} \cdot & \times & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$



$$\Gamma \propto |V_{us}|^2$$

$$\boxed{|V_{us}| = 0.2245 \pm 0.0008} \qquad (\approx \sin\theta_c)$$

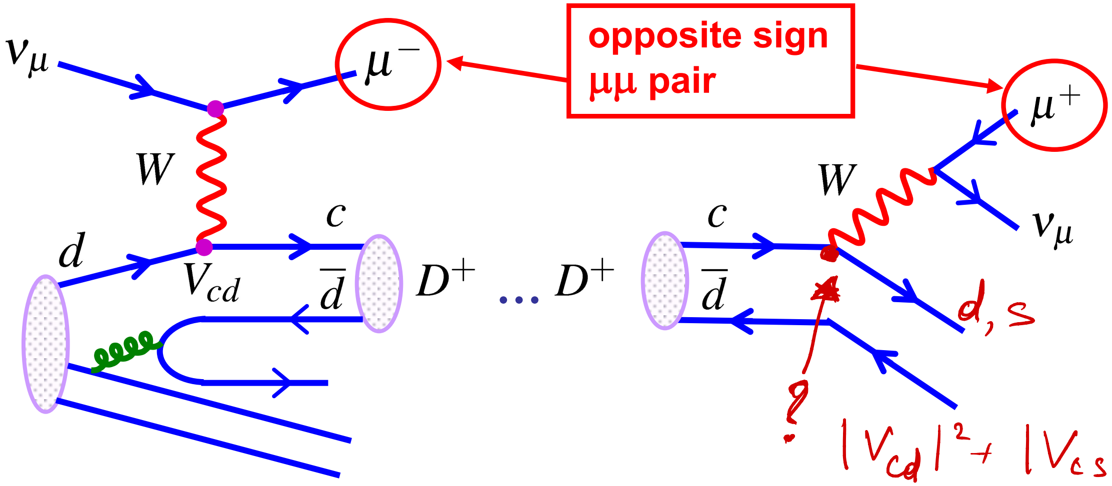
3 $|V_{cd}|$ from neutrino scattering; (semi-)leptonic D decays

$$\nu_\mu + N \to \mu^+\mu^- X$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \times & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Look for opposite charge di-muon events in ν_μ scattering from production and decay of a $D^+(c\bar{d})$ meson

$$\text{Rate} \propto |V_{cd}|^2 \underbrace{\text{Br}(D^+ \to X\mu^+\nu_\mu)}_{\text{Measured in various collider experiments}}$$



$$\Rightarrow \quad \boxed{|V_{cd}| = 0.221 \pm 0.004}$$

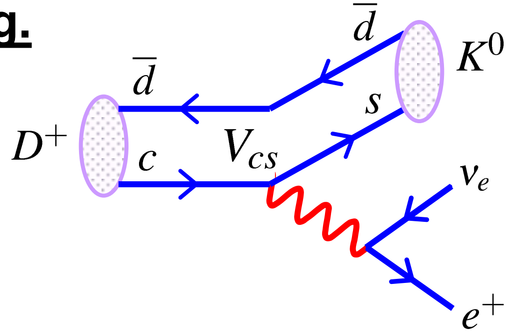
$|V_{cd}|^2 + |V_{cs}|^2 = 1 - |V_{cb}|^2$

contains this information

4 $|V_{cs}|$ from semi-leptonic charmed meson decays

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \times & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



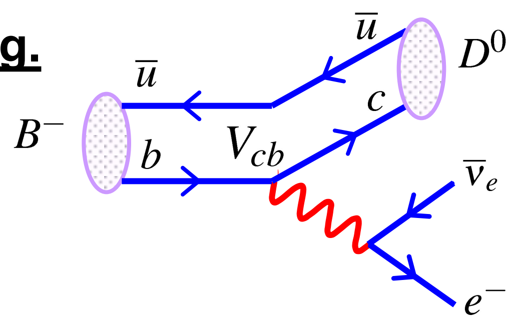
$$\Gamma \propto |V_{cs}|^2$$

$$\boxed{|V_{cs}| = 0.987 \pm 0.011}$$

5 $|V_{cb}|$ from semi-leptonic B hadron decays

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \times \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



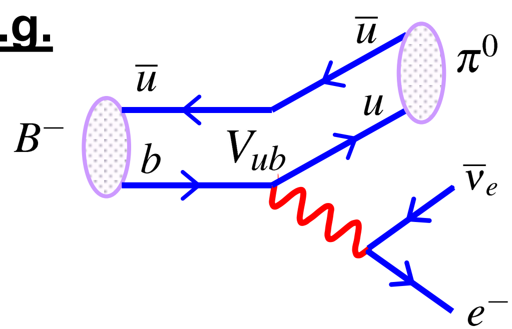
$$\Gamma \propto |V_{cb}|^2$$

$$\boxed{|V_{cb}| = (41.0 \pm 1.4) \times 10^{-3}}$$

6 $|V_{ub}|$ from semi-leptonic B hadron decays

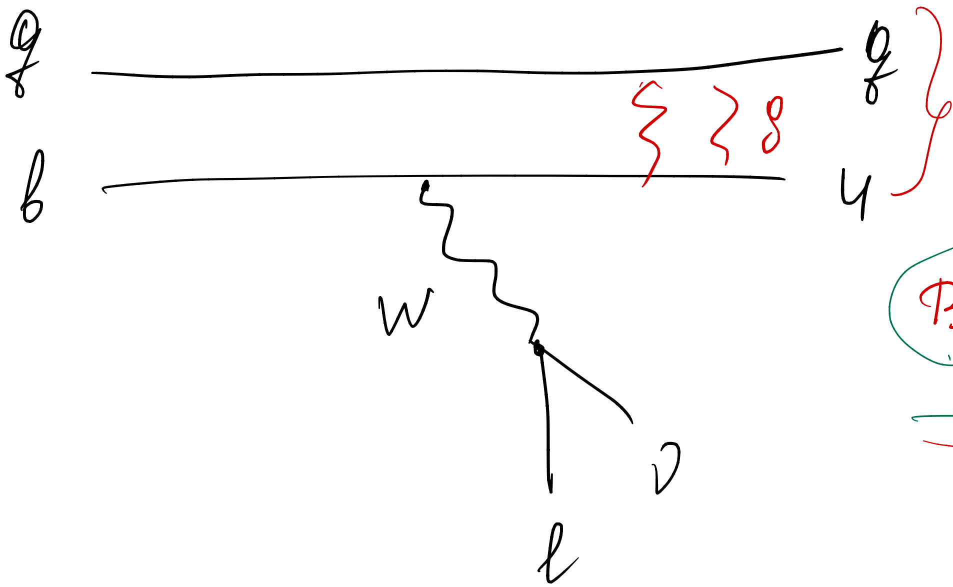
$$\begin{pmatrix} \cdot & \cdot & \times \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

e.g.



$$\Gamma \propto |V_{ub}|^2$$

$$\boxed{|V_{ub}| = (3.82 \pm 0.24) \times 10^{-3}}$$



Inclusive vs. exclusive

$\downarrow$ $\qquad\qquad$ $\uparrow$

$\underline{B^- \to (\pi^0/\rho/\eta)\,\ell\nu}$ $\qquad\qquad$ $\underline{B^- \to \pi^0 \ell\nu}$

$\uparrow$ $\qquad\qquad$ $\downarrow$

B- factories $\qquad\qquad$ V_{ub}

$$\underbrace{\underset{3\text{ GeV}}{\xrightarrow{e^+}} \;\cdot\; \underset{8\text{ GeV}}{\xleftarrow{e^-}}}_{11\text{ GeV}} \qquad \sqrt{s} = m(\Upsilon(4S))$$

$\downarrow$

$B^+ B^-$

$B^0 \bar{B}^0$

$(E_0, \vec{p}_0)$

$\boxed{V_{ub} \text{ puzzle}}$ ← incl. and exclusive are in tension

~~cb~~

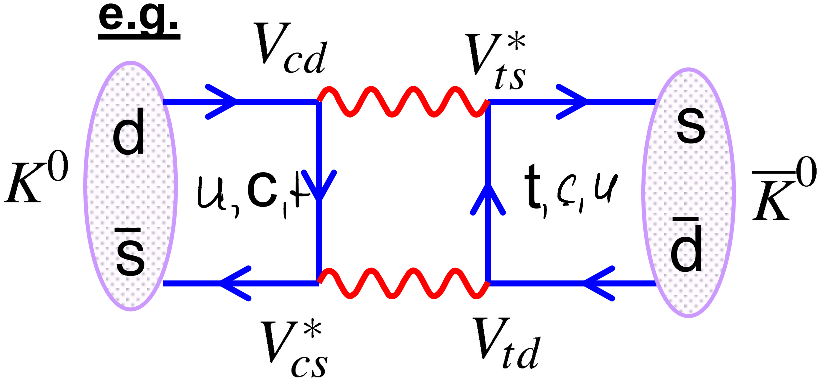
7/8 $|V_{td}|$ and $|V_{ts}|$

from K or B oscillations measurements

Δm_d : $B^0 - \overline{B^0}$

Δm_s : $B_s - \overline{B_s}$

e.g.



- E.g. ratio $|V_{td}/V_{ts}|$ determined from $\Delta m_d/\Delta m_s$ (mass difference in B^0 and in B_s systems)
- In ratios many uncertainties cancel out
- Precision limited by theoretical uncertainties

$$\boxed{|V_{td}/V_{ts}| = 0.205 \underbrace{\pm 0.001}_{\text{experimental error}} \underbrace{\pm 0.006}_{\text{theory uncertainty}}}$$

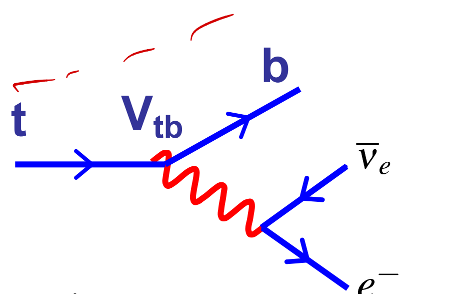
- Compare with uncertainties here:

$$|V_{td}| = (8.0 \pm 0.3) \times 10^{-3}\,, \qquad |V_{ts}| = (38.8 \pm 1.1) \times 10^{-3}$$

9 $|V_{tb}|$

from from top quark decays

e.g.



$t \to W s$
$\quad W d$ | ← upper limits

- Use ratio of BF: $B(t \to Wb)/B(t \to Wq) \propto |V_{tb}|^2$
- Also use single top quark production

$$\boxed{|V_{tb}| = 1.013 \pm 0.030}$$

- Experimental uncertainties dominate

$(t\bar{F}) - 2t$

0^-

5.28

Particle – Anti-Particle Mixing

- The wave-function for a single particle with lifetime $\tau = 1/\Gamma$ evolves with time as:

$$\psi(t) = Ne^{-\Gamma t/2}e^{-iMt}$$

which gives the appropriate exponential decay of

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\psi(0)\rangle e^{-t/\tau}$$

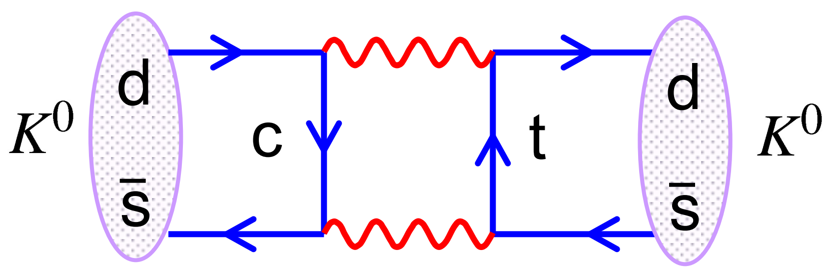
- The wave-function satisfies the time-dependent wave equation:

$$\hat{H}|\psi(t)\rangle = \boxed{(M - \tfrac{1}{2}i\Gamma)}|\psi(t)\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle \qquad \textbf{(A1)}$$

- For a bound state such as a K^0 the mass term includes the "mass" from the weak interaction "potential" $\hat{H}_{\mathrm{weak}}$

$$M = m_{K^0} + \langle K^0|\hat{H}_{\mathrm{weak}}|K^0\rangle + \sum_j \frac{|\langle K^0|\hat{H}_{\mathrm{weak}}|j\rangle|^2}{m_{K^0} - E_j}$$

Sum over intermediate states j



The third term is the 2nd order term in the perturbation expansion corresponding to box diagrams resulting in $K^0 \rightarrow \overline{K}^0$

- The total decay rate is the sum over all possible decays $K^0 \to f$

$$\Gamma = 2\pi \sum_f |\langle f|\hat{H}_{weak}|K^0\rangle|^2 \rho_F$$

Density of final states

★ Because there are also diagrams which allow $K^0 \leftrightarrow \overline{K}^0$ mixing need to consider the time evolution of a mixed state

$$\psi(t) = a(t)K^0 + b(t)\overline{K}^0 \qquad \text{(A2)}$$

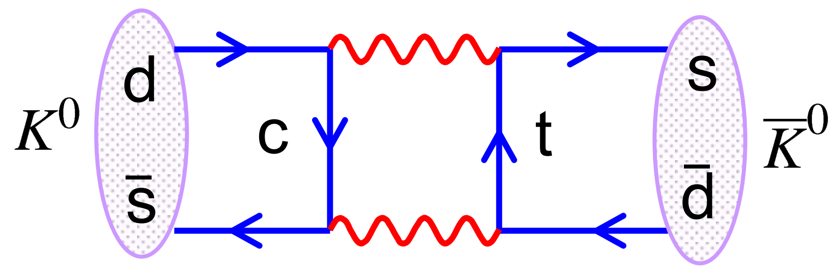
★ The time dependent wave-equation of (A1) becomes

$$\begin{pmatrix} M_{11} - \frac{1}{2}i\Gamma_{11} & M_{12} - \frac{1}{2}i\Gamma_{12} \\ M_{21} - \frac{1}{2}i\Gamma_{21} & M_{22} - \frac{1}{2}i\Gamma_{22} \end{pmatrix} \begin{pmatrix} |K^0(t)\rangle \\ |\overline{K}^0(t)\rangle \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} |K^0(t)\rangle \\ |\overline{K}^0(t)\rangle \end{pmatrix} \qquad \text{(A3)}$$

the diagonal terms are as before, and the off-diagonal terms are due to mixing.

$$M_{11} = m_{K^0} + \langle K^0|\hat{H}_{\text{weak}}|K^0\rangle + \sum_n \frac{|\langle K^0|\hat{H}_{\text{weak}}|K^0\rangle|^2}{m_{K^0} - E_n}$$

$$M_{12} = \sum_j \frac{\langle K^0|\hat{H}_{\text{weak}}|j\rangle^* \langle j|\hat{H}_{\text{weak}}|\overline{K}^0\rangle}{m_{K^0} - E_j}$$



- The off-diagonal decay terms include the effects of interference between decays to a common final state

$$\Gamma_{12} = 2\pi \sum_f \langle f|\hat{H}_{weak}|K^0\rangle^* \langle f|\hat{H}_{weak}|\overline{K}^0\rangle \rho_F$$

- In terms of the time dependent coefficients for the kaon states, (A3) becomes

$$\left[\mathbf{M} - i\tfrac{1}{2}\Gamma\right] \begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix}$$

where the Hamiltonian can be written:

$$\mathbf{H} = \mathbf{M} - i\tfrac{1}{2}\Gamma = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}$$

- Both the mass and decay matrices represent observable quantities and are Hermitian

$$M_{11} = M_{11}^*, \quad M_{22} = M_{22}^*, \quad M_{12} = M_{21}^*$$

$$\Gamma_{11} = \Gamma_{11}^*, \quad \Gamma_{22} = \Gamma_{22}^*, \quad \Gamma_{12} = \Gamma_{21}^*$$

- Furthermore, if CPT is conserved then the masses and decay rates of the $\overline{K}^0$ and K^0 are identical:

$$M_{11} = M_{22} = M; \qquad \Gamma_{11} = \Gamma_{22} = \Gamma$$

- Hence the time evolution of the system can be written:

$$\boxed{\begin{pmatrix} M-\frac{1}{2}i\Gamma & M_{12}-\frac{1}{2}i\Gamma_{12} \\ M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*} & M-\frac{1}{2}i\Gamma \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix}} \qquad \textbf{(A4)}$$

- To solve the coupled differential equations for a(t) and b(t), first find the eigenstates of the Hamiltonian (the K_L and K_S) and then transform into this basis. The eigenvalue equation is:

$$\begin{pmatrix} M-\frac{1}{2}i\Gamma & M_{12}-\frac{1}{2}i\Gamma_{12} \\ M_{12}^{*}-\frac{1}{2}i\Gamma_{12}^{*} & M-\frac{1}{2}i\Gamma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \textbf{(A5)}$$

- Which has non-trivial solutions for

$$|\mathbf{H}-\lambda I| = 0$$

$$\Rightarrow \quad (M-\tfrac{1}{2}i\Gamma-\lambda)^2 - (M_{12}^{*}-\tfrac{1}{2}i\Gamma_{12}^{*})(M_{12}-\tfrac{1}{2}i\Gamma_{12}) = 0$$

with eigenvalues

$$\lambda = M-\tfrac{1}{2}i\Gamma \pm \sqrt{(M_{12}^{*}-\tfrac{1}{2}i\Gamma_{12}^{*})(M_{12}-\tfrac{1}{2}i\Gamma_{12})}$$

- The eigenstates can be obtained by substituting back into **(A5)**

$$(M - \tfrac{1}{2}i\Gamma)x_1 + (M_{12} - \tfrac{1}{2}i\Gamma_{12}) = (M - \tfrac{1}{2}i\Gamma \pm \sqrt{(M^*_{12} - \tfrac{1}{2}i\Gamma^*_{12})(M_{12} - \tfrac{1}{2}i\Gamma_{12})})x_1$$

$$\Rightarrow \qquad \frac{x_2}{x_1} = \pm\sqrt{\frac{M^*_{12} - \frac{1}{2}i\Gamma^*_{12}}{M_{12} - \frac{1}{2}i\Gamma_{12}}}$$

★ **Define**

$$\eta = \sqrt{\frac{M^*_{12} - \frac{1}{2}i\Gamma^*_{12}}{M_{12} - \frac{1}{2}i\Gamma_{12}}}$$

★ **Hence the normalised eigenstates are**

$$|K_\pm\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\begin{pmatrix}1\\ \pm\eta\end{pmatrix} = \frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle \pm \eta|\overline{K}^0\rangle)$$

★ **Note, in the limit where M_{12}, Γ_{12} are real, the eigenstates correspond to the CP eigenstates K_1 and K_2. Hence we can identify the general eigenstates as as the long and short lived neutral kaons:**

$$\boxed{|K_L\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle + \eta|\overline{K}^0\rangle)} \qquad \boxed{|K_S\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle - \eta|\overline{K}^0\rangle)}$$

★ **Substituting these states back into (A2):**

$$
\begin{aligned}
|\psi(t)\rangle &= a(t)|K^0\rangle + b(t)|\overline{K}^0\rangle \\
&= \sqrt{1+|\eta|^2}\left[\frac{a(t)}{2}(K_L+K_S)+\frac{b(t)}{2\eta}(K_L-K_S)\right] \\
&= \sqrt{1+|\eta|^2}\left[\left(\frac{a(t)}{2}+\frac{b(t)}{2\eta}\right)K_L+\left(\frac{a(t)}{2}-\frac{b(t)}{2\eta}\right)K_S\right] \\
&= \frac{\sqrt{1+|\eta|^2}}{2}\left[a_L(t)K_L+a_S(t)K_S\right]
\end{aligned}
$$

with

$$\boxed{a_L(t)\equiv a(t)+\frac{b(t)}{\eta}} \qquad \boxed{a_S(t)\equiv a(t)-\frac{b(t)}{\eta}}$$

★ **Now consider the time evolution of** $a_L(t)$

$$i\frac{\partial a_L}{\partial t} = i\frac{\partial a}{\partial t}+\frac{i}{\eta}\frac{\partial b}{\partial t}$$

★ **Which can be evaluated using (A4) for the time evolution of a(t) and b(t):**

$$
\begin{aligned}
i\frac{\partial a_L}{\partial t} &= \left[(M-\tfrac{1}{2}i\Gamma_{12})a+(M_{12}-\tfrac{1}{2}i\Gamma_{12})b\right]+\frac{1}{\eta}\left[(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)a+(M-\tfrac{1}{2}i\Gamma)b\right]\\
&= (M-\tfrac{1}{2}i\Gamma)\left(a+\frac{b}{\eta}\right)+(M_{12}-\tfrac{1}{2}i\Gamma_{12})b+\frac{1}{\eta}(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)a\\
&= (M-\tfrac{1}{2}i\Gamma)a_L+(M_{12}-\tfrac{1}{2}i\Gamma_{12})b+\left(\sqrt{(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right)a\\
&= (M-\tfrac{1}{2}i\Gamma)a_L+\left(\sqrt{(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right)\left(a+\frac{b}{\eta}\right)\\
&= (M-\tfrac{1}{2}i\Gamma)a_L+\left(\sqrt{(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right)a_L\\
&= (m_L-\tfrac{1}{2}i\Gamma_L)a_L
\end{aligned}
$$

★ **Hence:**

$$
\boxed{i\frac{\partial a_L}{\partial t}=(m_L-\tfrac{1}{2}i\Gamma_L)a_L}
$$

with $$m_L = M+\Re\left\{\sqrt{(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right\}$$

and $$\Gamma_L = \Gamma-2\Im\left\{\sqrt{(M_{12}^*-\tfrac{1}{2}i\Gamma_{12}^*)(M_{12}-\tfrac{1}{2}i\Gamma_{12})}\right\}$$

★ **Following the same procedure obtain:**

$$\boxed{i\frac{\partial a_S}{\partial t} = (m_S - \tfrac{1}{2}i\Gamma_S)a_S}$$

with $$m_S = M - \Re\left\{\sqrt{(M_{12}^* - \tfrac{1}{2}i\Gamma_{12}^*)(M_{12} - \tfrac{1}{2}i\Gamma_{12})}\right\}$$

and $$\Gamma_S = \Gamma + 2\Im\left\{\sqrt{(M_{12}^* - \tfrac{1}{2}i\Gamma_{12}^*)(M_{12} - \tfrac{1}{2}i\Gamma_{12})}\right\}$$

★ **In matrix notation we have**

$$\begin{pmatrix} M_L - \frac{1}{2}i\Gamma_L & 0 \\ 0 & M_S - \frac{1}{2}i\Gamma_S \end{pmatrix}\begin{pmatrix} a_L \\ a_S \end{pmatrix} = i\frac{\partial}{\partial t}\begin{pmatrix} a_L \\ a_S \end{pmatrix}$$

★ **Solving we obtain**

$$a_L(t) \propto e^{-im_Lt - \Gamma_Lt/2} \qquad a_S(t) \propto e^{-im_St - \Gamma_St/2}$$

★ **Hence in terms of the K_L and K_S basis the states propagate as independent particles with definite masses and lifetimes (the mass eigenstates). The time evolution of the neutral kaon system can be written**

$$|\psi(t)\rangle = A_Le^{-im_Lt - \Gamma_Lt/2}|K_L\rangle + A_Se^{-im_St - \Gamma_St/2}|K_S\rangle$$

where A_L and A_S are constants

CP Violation : $\pi\pi$ decays

★ Consider the development of the $K^0 - \overline{K}^0$ system now including CP violation

★ Repeat previous derivation using

$$|K_S\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}[|K_1\rangle + \varepsilon|K_2\rangle] \qquad |K_L\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}}[|K_2\rangle + \varepsilon|K_1\rangle]$$

• Writing the CP eigenstates in terms of $K^0, \overline{K}^0$

$$|K_L\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)|K_0\rangle + (1-\varepsilon)|\overline{K}^0\rangle\right]$$

$$|K_S\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)|K_0\rangle - (1-\varepsilon)|\overline{K}^0\rangle\right]$$

• Inverting these expressions obtain

$$|K^0\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1+\varepsilon}(|K_L\rangle + |K_S\rangle) \qquad |\overline{K}^0\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1-\varepsilon}(|K_L\rangle - |K_S\rangle)$$

• Hence a state that was produced as a K^0 evolves with time as:

$$|\psi(t)\rangle = \sqrt{\frac{1+|\varepsilon|^2}{2}}\frac{1}{1+\varepsilon}(\theta_L(t)|K_L\rangle + \theta_S(t)|K_S\rangle)$$

where as before $\theta_S(t) = e^{-(im_S+\frac{\Gamma_S}{2})t}$ and $\theta_L(t) = e^{-(im_L+\frac{\Gamma_L}{2})t}$

• If we are considering the decay rate to $\pi\pi$ need to express the wave-function in terms of the CP eigenstates (remember we are neglecting CP violation in the decay)

$$
\begin{aligned}
|\psi(t)\rangle &= \frac{1}{\sqrt{2}}\frac{1}{1+\varepsilon}\left[(|K_2\rangle+\varepsilon|K_1\rangle)\theta_L(t)+(|K_1\rangle+\varepsilon|K_2\rangle)\theta_S(t)\right] \\
&= \frac{1}{\sqrt{2}}\frac{1}{1+\varepsilon}\left[(\theta_S+\varepsilon\theta_L)|K_1\rangle+(\theta_L+\varepsilon\theta_S)|K_2\rangle\right]
\end{aligned}
$$

CP Eigenstates

• Two pion decays occur with CP = +1 and therefore arise from decay of the CP = +1 kaon eigenstate, i.e. K_1

$$\Gamma(K^0_{t=0}\to\pi\pi)\propto|\langle K_1|\psi(t)\rangle|^2=\frac{1}{2}\left|\frac{1}{1+\varepsilon}\right|^2|\theta_S+\varepsilon\theta_L|^2$$

• Since $|\varepsilon|\ll 1$

$$\left|\frac{1}{1+\varepsilon}\right|^2=\frac{1}{(1+\varepsilon^*)(1+\varepsilon)}\approx\frac{1}{1+2\Re\{\varepsilon\}}\approx 1-2\Re\{\varepsilon\}$$

• Now evaluate the $|\theta_S+\varepsilon\theta_L|^2$ term again using

$$|z_1\pm z_2|^2=|z_1|^2+|z_2|^2\pm 2\Re(z_1z_2^*)$$

$$
\begin{aligned}
|\theta_S + \varepsilon\theta_L|^2 &= |e^{-im_S t - \frac{\Gamma_S}{2}t} + \varepsilon e^{-im_L t - \frac{\Gamma_L}{2}t}|^2 \\
&= e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2\Re\{e^{-im_S t - \frac{\Gamma_S}{2}t}.\varepsilon^* e^{+im_L t - \frac{\Gamma_L}{2}t}\}
\end{aligned}
$$

- **Writing** $\varepsilon = |\varepsilon| e^{i\phi}$

$$
\begin{aligned}
|\theta_S + \varepsilon\theta_L|^2 &= e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2}\Re\{e^{i(m_L - m_S)t - \phi}\} \\
&= e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2}\cos(\Delta m.t - \phi)
\end{aligned}
$$

- **Putting this together we obtain:**

$$
\boxed{\Gamma(K^0_{t=0} \to \pi\pi) = \frac{1}{2}(1 - 2\Re\{\varepsilon\})N_{\pi\pi}\left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} + 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2}\cos(\Delta m.t - \phi)\right]}
$$

Short lifetime component $K_S \to \pi\pi$

CP violating long lifetime component $K_L \to \pi\pi$

Interference term

- **In exactly the same manner obtain for a beam which was produced as** $\overline{K}^0$

$$
\boxed{\Gamma(\overline{K}^0_{t=0} \to \pi\pi) = \frac{1}{2}(1 + 2\Re\{\varepsilon\})N_{\pi\pi}\left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t} - 2|\varepsilon| e^{-(\Gamma_S+\Gamma_L)t/2}\cos(\Delta m.t - \phi)\right]}
$$

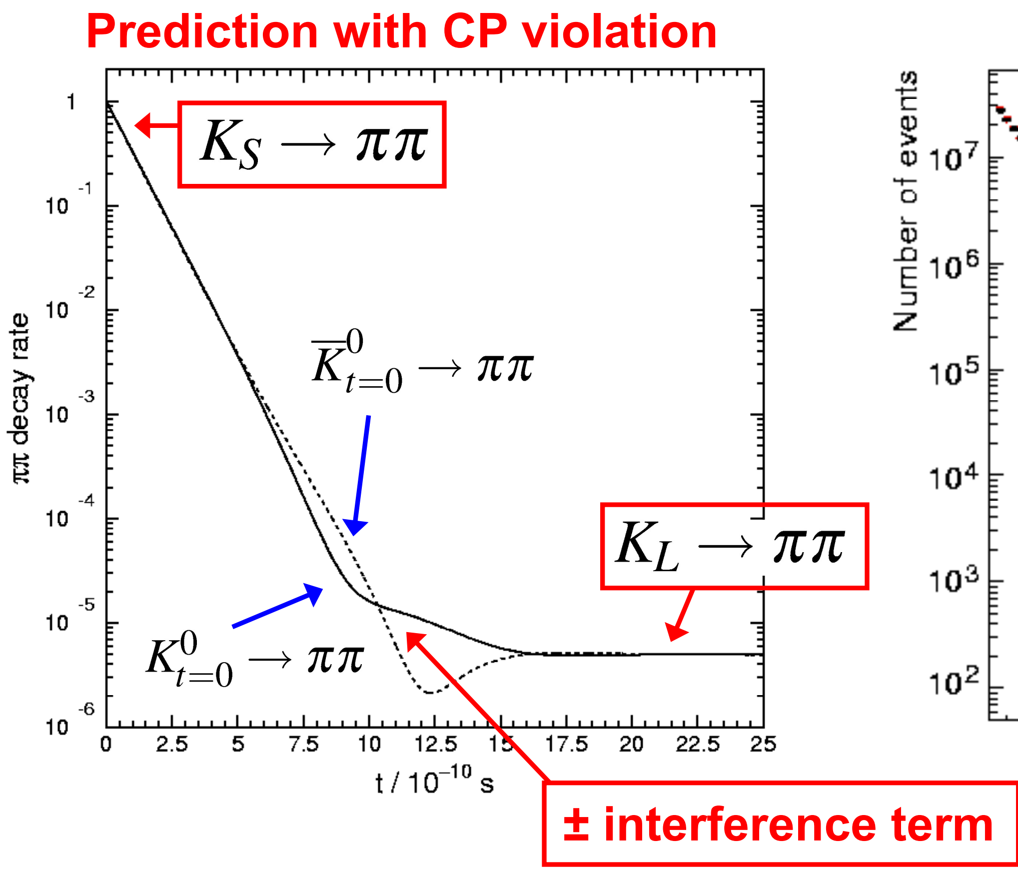
Interference term changes sign

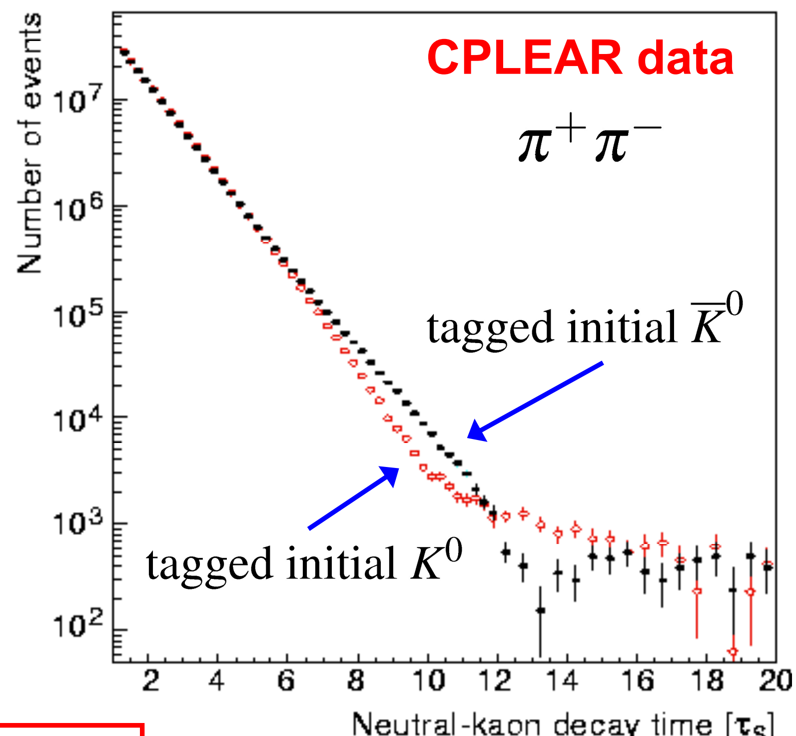
★ **At large proper times only the long lifetime component remains :**

$$\Gamma(K^0_{t=0} \to \pi\pi) \quad \to \quad \frac{1}{2}(1 - 2\Re\{\varepsilon\})N_{\pi\pi}.|\varepsilon|^2 e^{-\Gamma_L t}$$

i.e. CP violating $K_L \to \pi\pi$ **decays**

★ **Since CPLEAR can identify whether a** K^0 **or** $\overline{K}^0$ **was produced, able to measure** $\Gamma(K^0_{t=0} \to \pi\pi)$ **and** $\Gamma(\overline{K}^0_{t=0} \to \pi\pi)$





★ **The CPLEAR data shown previously can be used to measure** $\varepsilon = |\varepsilon| e^{i\phi}$

- **Define the asymmetry:**

$$A_{+-} = \frac{\Gamma(\overline{K}^0_{t=0} \to \pi\pi) - \Gamma(K^0_{t=0} \to \pi\pi)}{\Gamma(\overline{K}^0_{t=0} \to \pi\pi) + \Gamma(K^0_{t=0} \to \pi\pi)}$$

- **Using previous expressions:**

$$A_{+-} = \frac{4\Re\{\varepsilon\}\left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}\right] - 4|\varepsilon| e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t - \phi)}{2\left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}\right] - \underbrace{8\Re\{\varepsilon\}|\varepsilon| e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t - \phi)}}$$

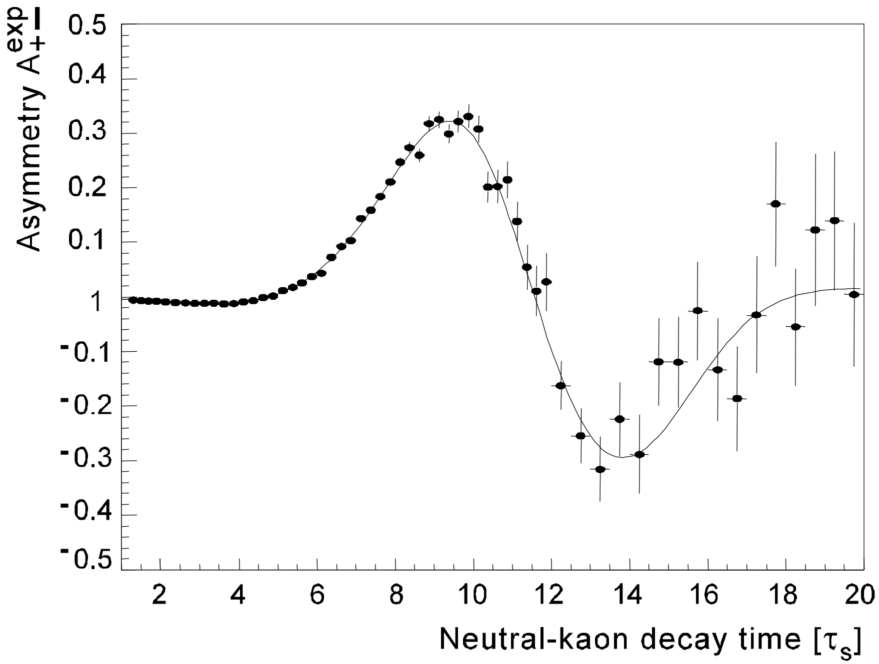
$\propto |\varepsilon|\Re\{\varepsilon\}$ **i.e. two small quantities and can safely be neglected**

$$A_{+-} \approx \frac{2\Re\{\varepsilon\}\left[e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}\right] - 2|\varepsilon| e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t - \phi)}{e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}}$$

$$= 2\Re\{\varepsilon\} - \frac{2|\varepsilon| e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m.t - \phi)}{e^{-\Gamma_S t} + |\varepsilon|^2 e^{-\Gamma_L t}}$$

$$= 2\Re\{\varepsilon\} - \frac{2|\varepsilon| e^{(\Gamma_S-\Gamma_L)t/2}\cos(\Delta m.t - \phi)}{1 + |\varepsilon|^2 e^{(\Gamma_S-\Gamma_L)t}}$$

A.Apostolakis et al., Eur. Phys. J. C18 (2000) 41



Best fit to the data:

$$|\varepsilon| = (2.264 \pm 0.035) \times 10^{-3}$$
$$\phi = (43.19 \pm 0.73)^\circ$$

CP Violation via Mixing

★ A full description of the SM origin of CP violation in the kaon system is beyond the level of this course, nevertheless, the relation to the box diagrams is illustrated below

★ The K-long and K-short wave-functions depend on η

$$\boxed{|K_L\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle + \eta|\overline{K}^0\rangle)} \quad \boxed{|K_S\rangle = \frac{1}{\sqrt{1+|\eta|^2}}(|K^0\rangle - \eta|\overline{K}^0\rangle)}$$

with

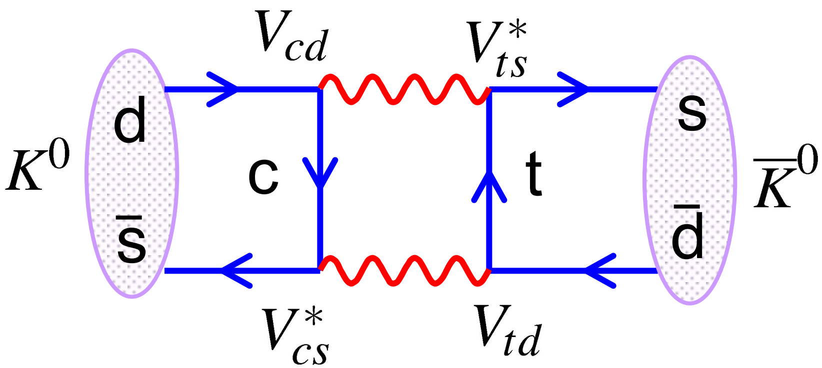
$$\eta = \sqrt{\frac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}}$$

★ If $M_{12}^* = M_{12}; \quad \Gamma_{12}^* = \Gamma_{12}$ then the K-long and K-short correspond to the CP eigenstates K_1 and K_2

- CP violation is therefore associated with imaginary off-diagonal mass and decay elements for the neutral kaon system
- Experimentally, CP violation is small and $\eta \approx 1$
- Define: $\varepsilon = \frac{1-\eta}{1+\eta} \quad \Rightarrow \quad \eta = \frac{1-\varepsilon}{1+\varepsilon}$

- Consider the mixing term M_{12} which arises from the sum over all possible intermediate states in the mixing box diagrams

e.g.



$$M_{12} = A_{ct} V_{cd} V_{cs}^* V_{ts}^* V_{td} + \ldots$$

- Therefore it can be seen that, in the Standard Model, CP violation is associated with the imaginary components of the CKM matrix

- It can be shown that mixing leads to CP violation with

$$|\varepsilon| \propto \Im\{M_{12}\}$$

- The differences in masses of the mass eigenstates can be shown to be:

$$\Delta m_K = m_{K_L} - m_{K_S} \approx \sum_{q,q'} \frac{G_F^2}{3\pi^2} f_K^2 m_K |V_{qd} V_{qs}^* V_{q'd} V_{q's}^*| m_q m_{q'}$$

where q and q' are the quarks in the loops and f_K is a constant

• In terms of the small parameter ε

$$|K_L\rangle = \frac{1}{2\sqrt{1+|\varepsilon|^2}}\left[(1+\varepsilon)|K^0\rangle + (1-\varepsilon)|\overline{K}^0\rangle\right]$$

$$|K_S\rangle = \frac{1}{2\sqrt{1+|\varepsilon|^2}}\left[(1-\varepsilon)|K^0\rangle + (1+\varepsilon)|\overline{K}^0\rangle\right]$$

★ If epsilon is non-zero we have CP violation in the neutral kaon system

Writing $\quad \eta = \sqrt{\dfrac{M_{12}^* - \frac{1}{2}i\Gamma_{12}^*}{M_{12} - \frac{1}{2}i\Gamma_{12}}} = \sqrt{\dfrac{z^*}{z}}$ $\quad$ and $\quad z = ae^{i\phi}$

gives $\quad \eta = e^{-i\phi}$

★ From which we can find an expression for ε

$$\varepsilon.\varepsilon^* = \frac{1-e^{-i\phi}}{1+e^{-i\phi}}.\frac{1-e^{+i\phi}}{1+e^{i\phi}} = \frac{2-\cos\phi}{2+\cos\phi} = \tan^2\frac{\phi}{2}$$

$$|\varepsilon| = |\tan\tfrac{\phi}{2}|$$

★ Experimentally we know ε is small, hence ϕ is small

$$|\varepsilon| \approx \frac{1}{2}\phi = \frac{1}{2}\arg z \approx \frac{1}{2}\frac{\Im\{M_{12} - \frac{1}{2}i\Gamma_{12}\}}{|M_{12} - \frac{1}{2}i\Gamma_{12}|}$$

Time Reversal Violation

- Previously obtained expressions for strangeness oscillations in the absence of CP violation, e.g.

$$\Gamma(K^0_{t=0} \to K^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

- This analysis can be extended to include the effects of CP violation to give the following rates:

$$\Gamma(K^0_{t=0} \to K^0) \propto \tfrac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma(\overline{K}^0_{t=0} \to \overline{K}^0) \propto \tfrac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma(\overline{K}^0_{t=0} \to K^0) \propto \tfrac{1}{4}\boxed{(1+4\Re\{\varepsilon\})}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

$$\Gamma(K^0_{t=0} \to \overline{K}^0) \propto \tfrac{1}{4}\boxed{(1-4\Re\{\varepsilon\})}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-(\Gamma_S+\Gamma_L)t/2}\cos\Delta m t\right]$$

★ Including the effects of CP violation find that

$$\boxed{\Gamma(\overline{K}^0_{t=0} \to K^0) \neq \Gamma(K^0_{t=0} \to \overline{K}^0)}$$

Violation of time reversal symmetry !

★ No surprise, as CPT is conserved, CP violation implies T violation

B-physics

- The oscillation of neutral mesons have been observed for heavy mesons as well:

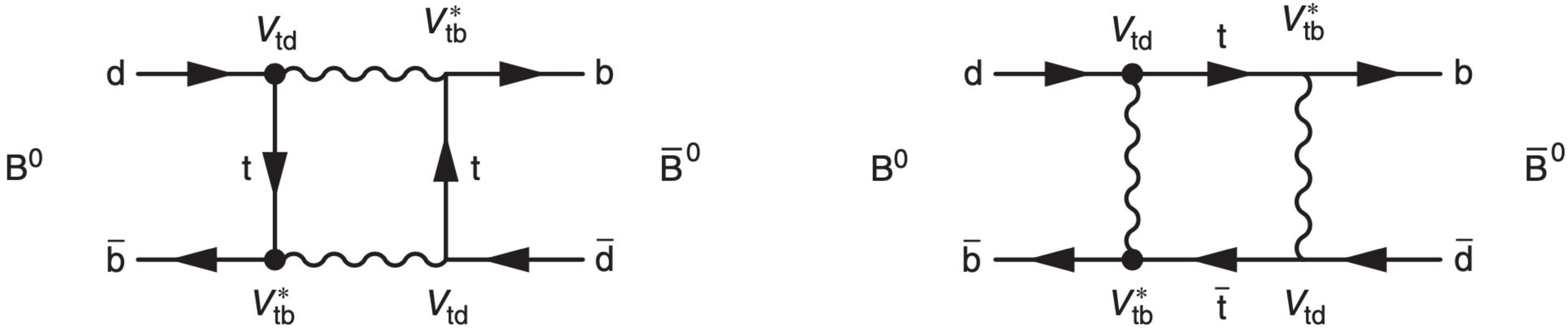
$$B^0(\bar{b}d) \leftrightarrow \overline{B}^0(b\bar{d}), \quad B_s^0(\bar{b}s) \leftrightarrow \overline{B}_s^0(b\bar{s}) \quad \text{and} \quad D^0(\bar{c}u) \leftrightarrow \overline{D}^0(c\bar{u})$$

- The studies of B^0 mesons by BaBar, Belle and LHCb experiments, and of B_s mesons by the LHCb provided crucial information on the CKM matrix and CP violation
- Mathematical description of B-oscillations is the same as for kaons
- Major difference:
  - as B-mesons are heavy they have many decay modes (few hundreds)
  - Typically the decays for B0 and B0-bar are different
  - Hence the interference between the decays is small
  - Consequence: oscillations can be described by a single angle β

$$|B_L\rangle = \frac{1}{\sqrt{2}}\left[|B^0\rangle + e^{-i2\beta}|\overline{B}^0\rangle\right] \quad \text{and} \quad |B_H\rangle = \frac{1}{\sqrt{2}}\left[|B^0\rangle - e^{-i2\beta}|\overline{B}^0\rangle\right]$$

B_L and B_H are a lighter and a heavier states with almost identical lifetimes

B^0 mixing



- In kaon mixing, the contributions from all quarks to a box diagram were similar
- Here, since $|V_{tb}| \gg |V_{ts}| > |V_{td}|$, only box diagrams with top quark matter
- Off-diagonal mass element $M_{12} \propto (V_{td}V_{tb}^*)^2$
- Mass difference $\Delta m_d = m(B_H) - m(B_L) = 2|M_{12}| \propto |(V_{td}V_{tb}^*)^2|$
- Since $|V_{tb}| \approx 1$, Δm_d allows to measure V_{td}

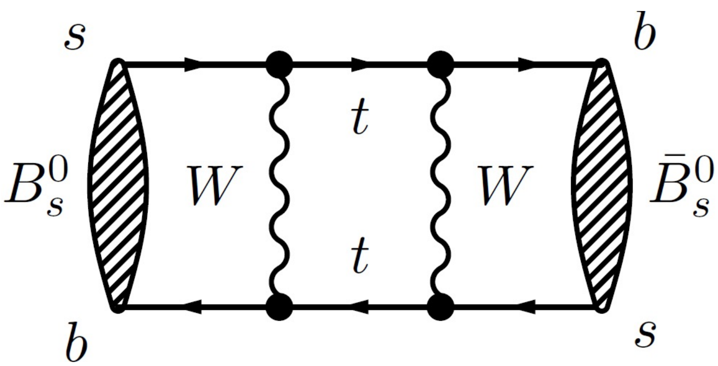
$$\Delta m_\mathrm{d} = (0.507 \pm 0.005)\,\mathrm{ps}^{-1} \equiv (3.34 \pm 0.03) \times 10^{-13}\,\mathrm{GeV}$$

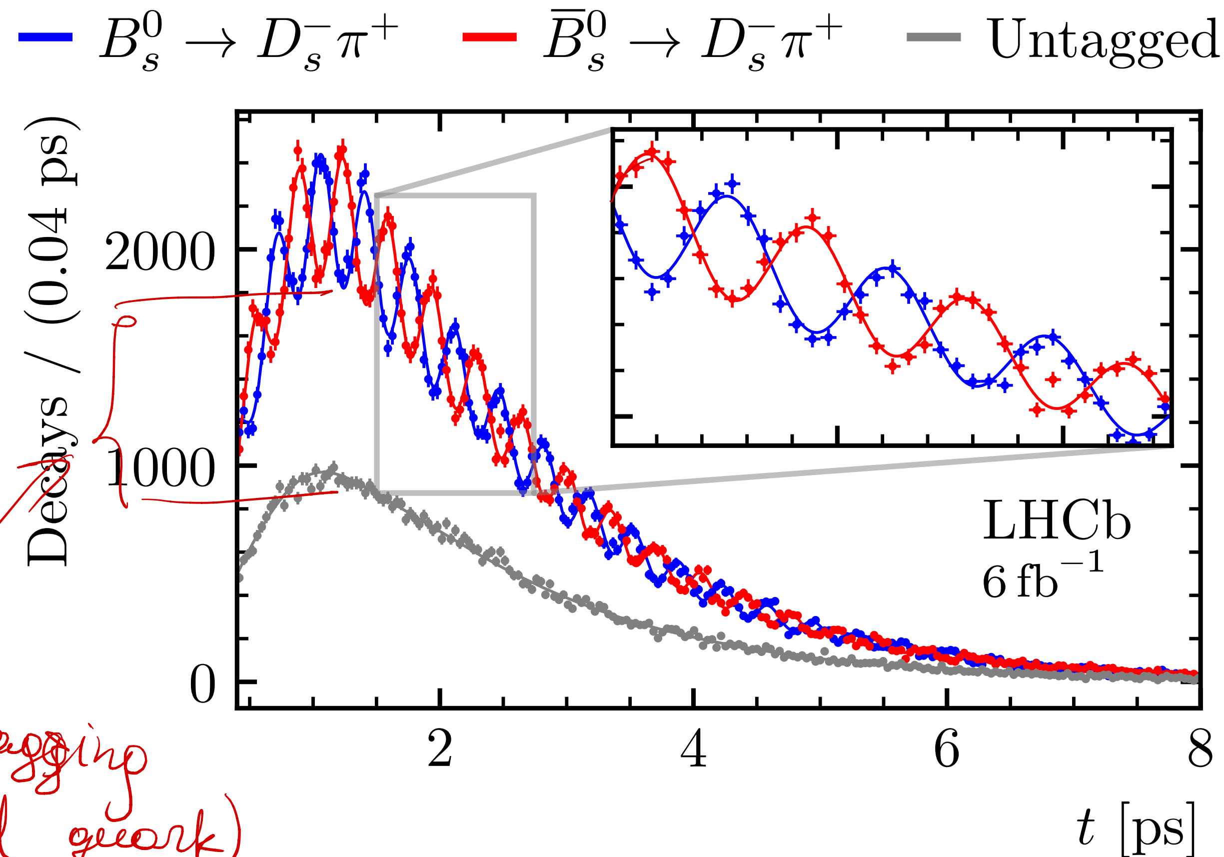
- Similarly, V_{ts} is derived from Δm_s in B_s oscillations

$$\Delta m_s = 17.7656 \pm 0.0057\,\mathrm{ps}^{-1}$$

B_s mixing

- **The latest LHCb result: https://arxiv.org/abs/2104.04421**





mistag (wrong tagging of second quark)

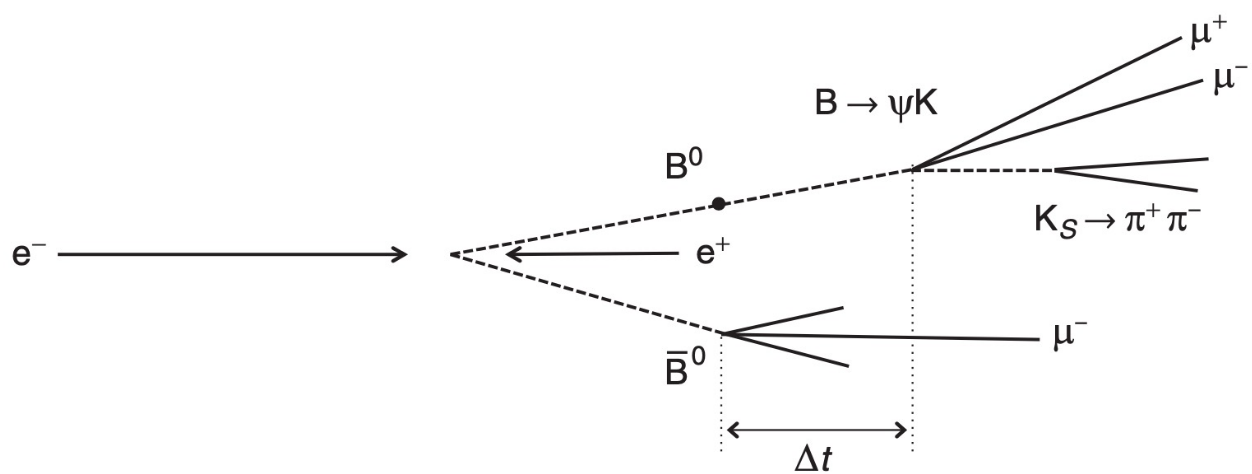
$$\Delta m_s = 17.7656 \pm 0.0057 \text{ps}^{-1}$$

CP violation in B-meson system

- CP violation is observed as three effects:
  - Direct CP violation in decay: $\Gamma(A \to X) \neq \Gamma(\text{A-bar} \to \text{X-bar})$, as parameterised by ε' in the neutral kaon system;
  - CP violation in the mixing of neutral mesons as parameterised by ε in the kaon system;
  - CP violation in the interference between decays to a common final state f with and without mixing, for example $B^0 \to f$ and $B^0 \to B^0\text{-bar} \to f$.
- In the SM, the CP violation in B^0 mixing is small
- CP-violating effects in the interference between decays $B^0 \to f$ and $B^0 \to B^0\text{-bar} \to f$ can be relatively large and have been studied extensively by the BaBar, Belle and LHCb experiments

CP violation in the interference: $B^0 \to J/\psi K_S$

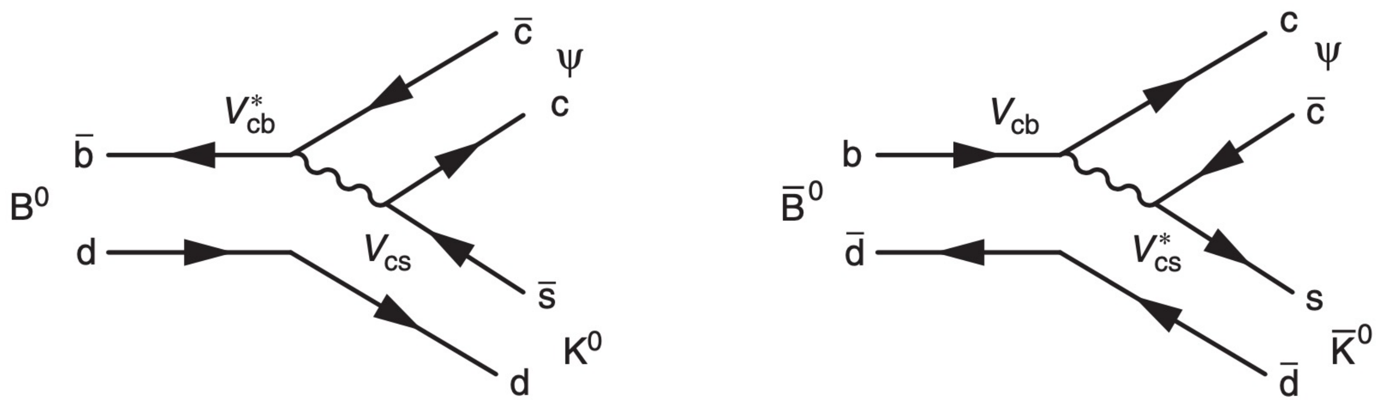
- J/ψ: $J^P = 1^-$, CP = +1
- K_S: $J^P = 0^-$, CP = +1
- B^0 are spin-0 ⇒ $B^0 \to J/\psi K_S$ has L=1 orbital angular momentum
- $CP(J/\psi K_S) = CP(J/\psi)CP(K_S)(-1)^L = (+1)(+1)(-1) = -1$
- Similarly, $B^0 \to J/\psi K_L$ occurs in the CP-even state



- Muon charge from B^0-bar $\to D^+ \mu^- \nu_\mu$-bar tags a flavour of B^0-bar, and a second B^0 at the moment t=0
- Decay to $J/\psi K_S$ happens either directly or after mixing

CP violation in the interference: $B^0 \rightarrow J/\psi K_S$

- **Decay proceeds in two stages:**
  - **First, B^0 decays to flavour eigenstate: $B^0 \rightarrow J/\psi K^0$ and B^0-bar$\rightarrow J/\psi K^0$-bar**
  - **Then the neutral kaons system evolves as a linear combination of physical K_S and K_L states**

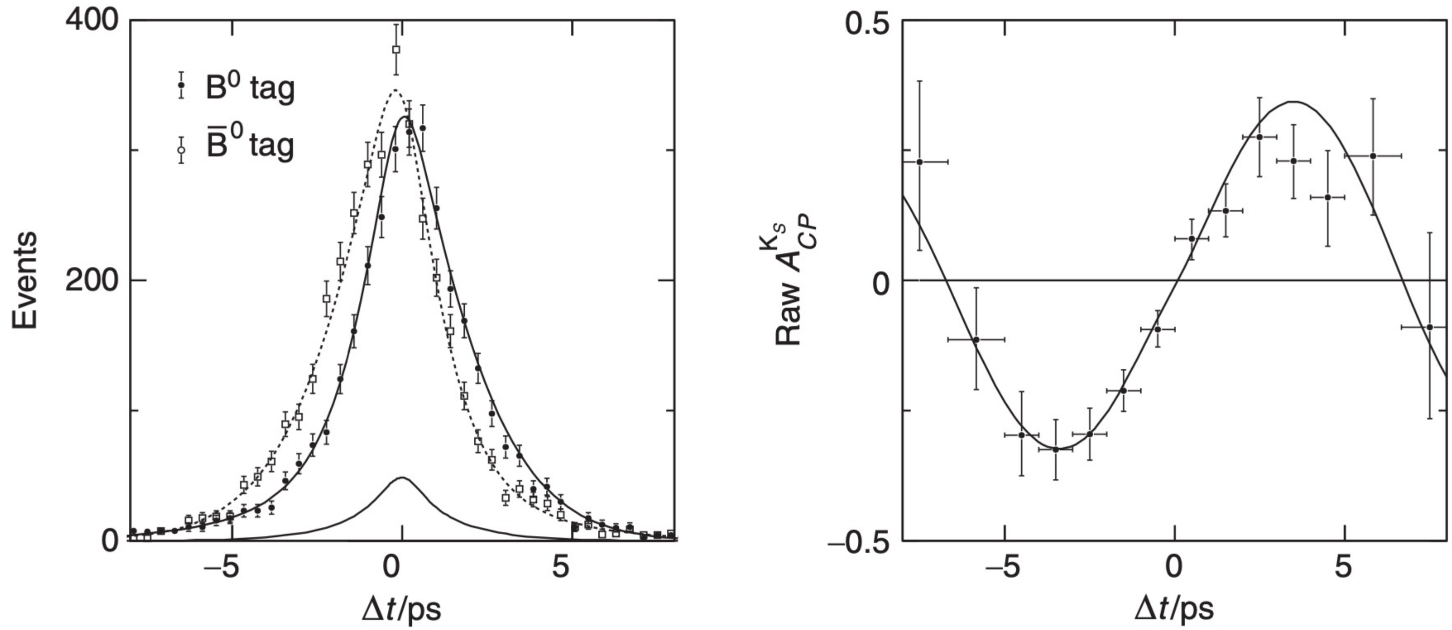


- **CP violation is measurable through the asymmetry**

$$A_{CP}^{K_S} = \frac{\Gamma(\overline{B}^0_{t=0} \rightarrow \psi K_S) - \Gamma(B^0_{t=0} \rightarrow \psi K_S)}{\Gamma(\overline{B}^0_{t=0} \rightarrow \psi K_S) + \Gamma(B^0_{t=0} \rightarrow \psi K_S)} = \sin(\Delta m_d t)\ \sin(2\beta).$$

CP violation in the interference: $B^0 \to J/\psi K_S$

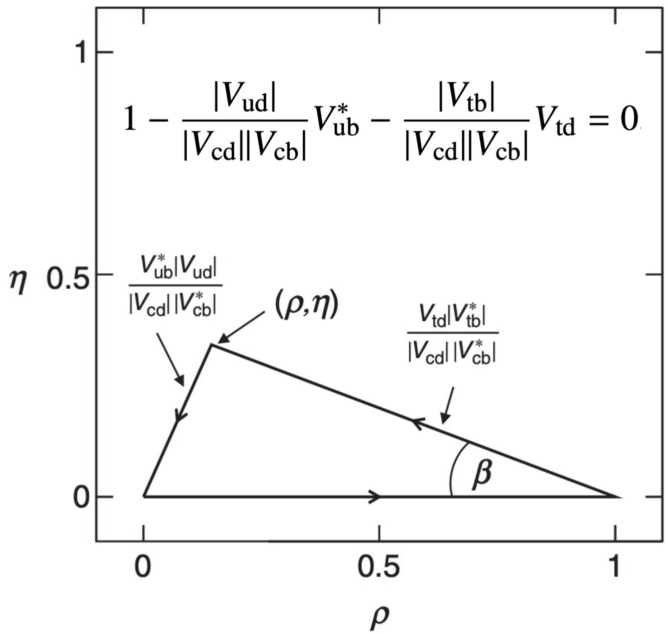
- Results from Belle:

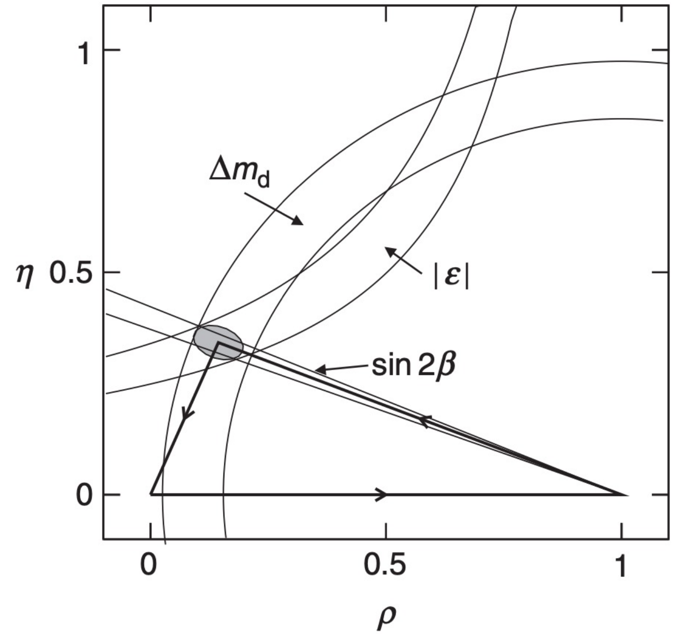


- the observation of a non-zero value of sin(2β) is a direct manifestation of CP violation in the B-meson system
- Measured value: sin(2β) = 0.685 ± 0.032

CP violation in the quark sector

- Combination of various measurements:





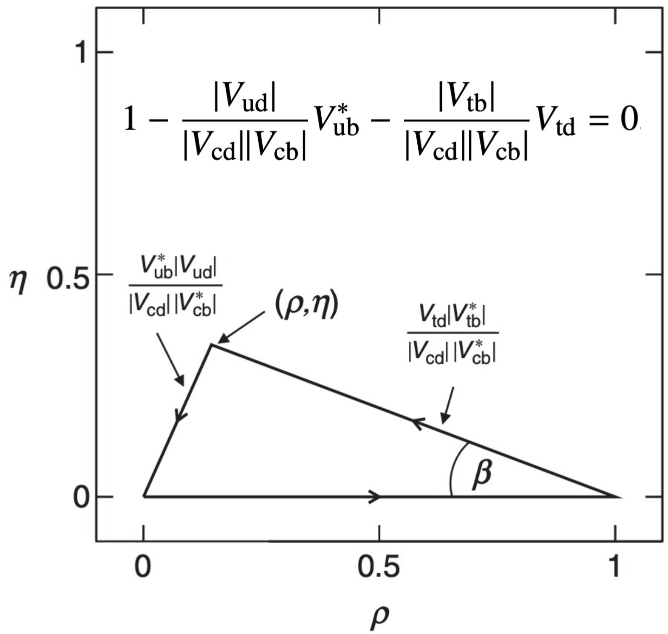
- CP violation in the weak interactions of hadrons is described by the single irreducible complex phase in the CKM matrix
- In the Wolfenstein parametrisation, CP violation is associated with the parameter η. To O(λ^4), the parameter η appears only in V_{ub} and V_{td}, with

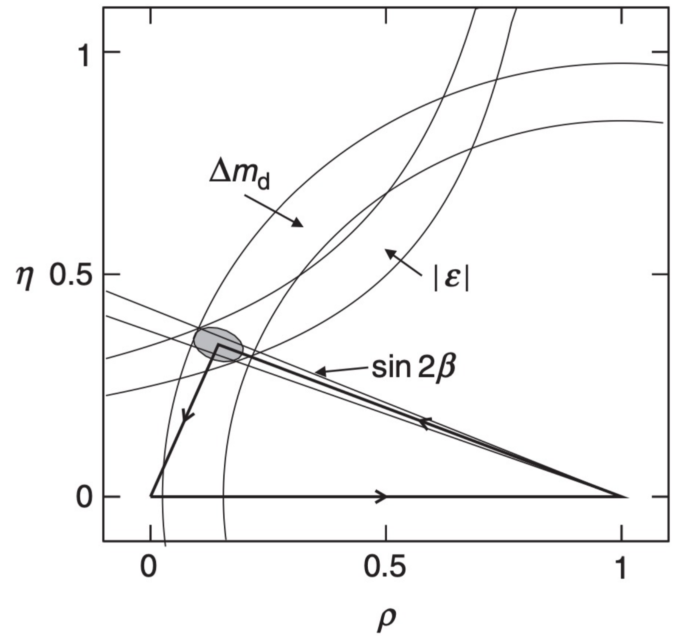
$$V_{ub} \approx A\lambda^3(\rho - i\eta) \quad \text{and} \quad V_{td} \approx A\lambda^3(1 - \rho - i\eta)$$

- The measurements of non-zero values of |ε| and sin(2β) separately imply that η ≠ 0
- But ρ and η are determined from the measurements combinations

CP violation in the quark sector

- **Combination of various measurements:**

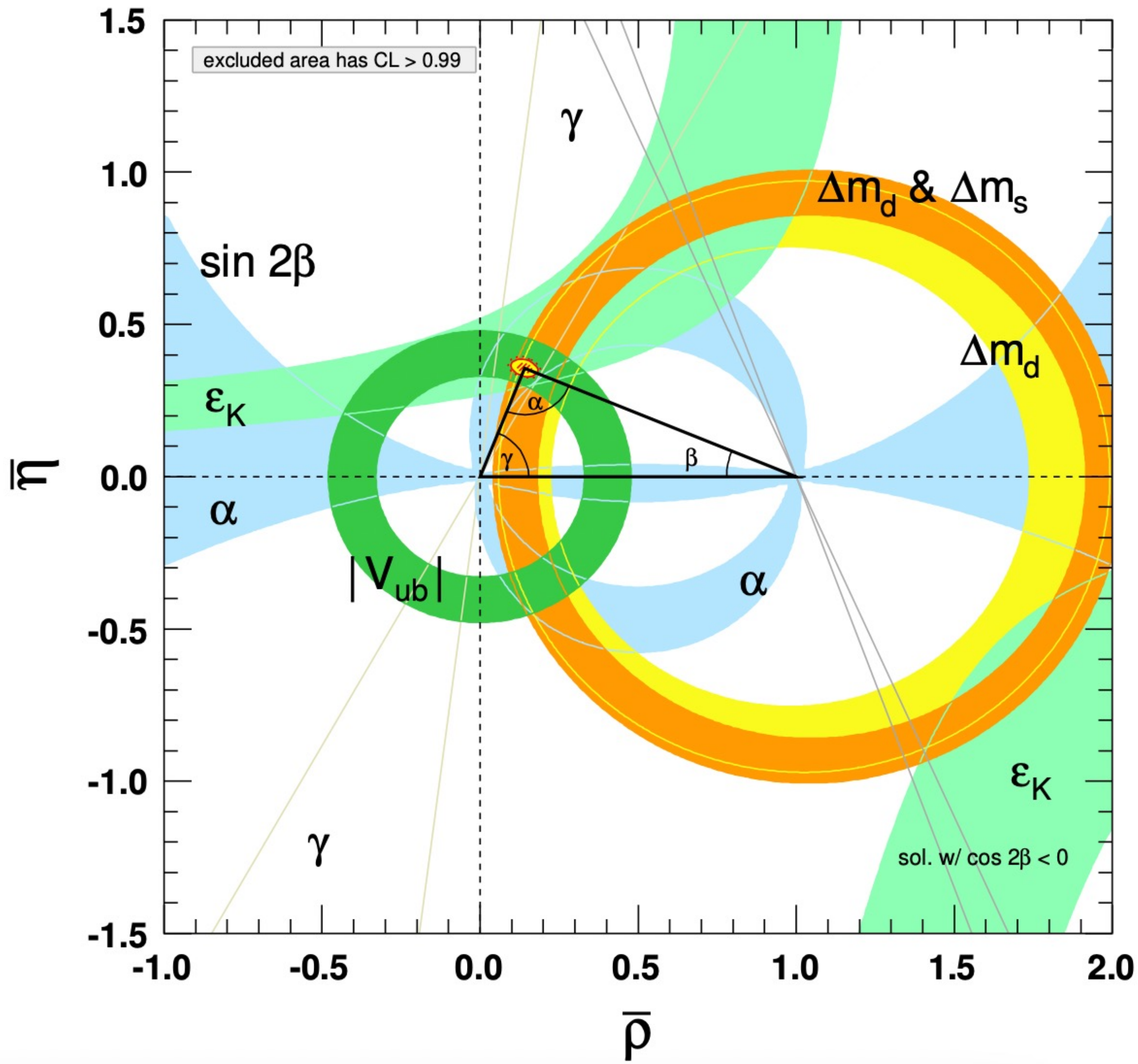




$$\lambda = 0.2253 \pm 0.0007,\ A = 0.811^{+0.022}_{-0.012},\ \rho = 0.13 \pm 0.02,\ \eta = 0.345 \pm 0.014.$$

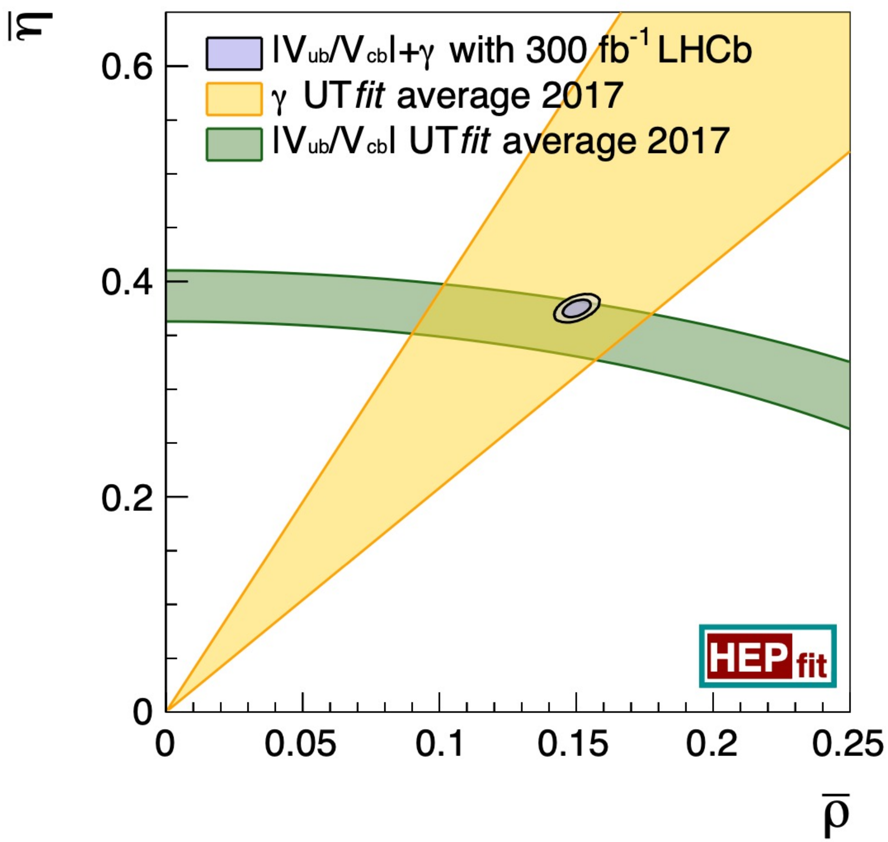
- **With all the measurements, the unitary triangle is closed**
- **A deviation from this could indicate a presence of a new physics**
- **Future experiments, LHCb in particular, will improve many measurements by orders of magnitude**

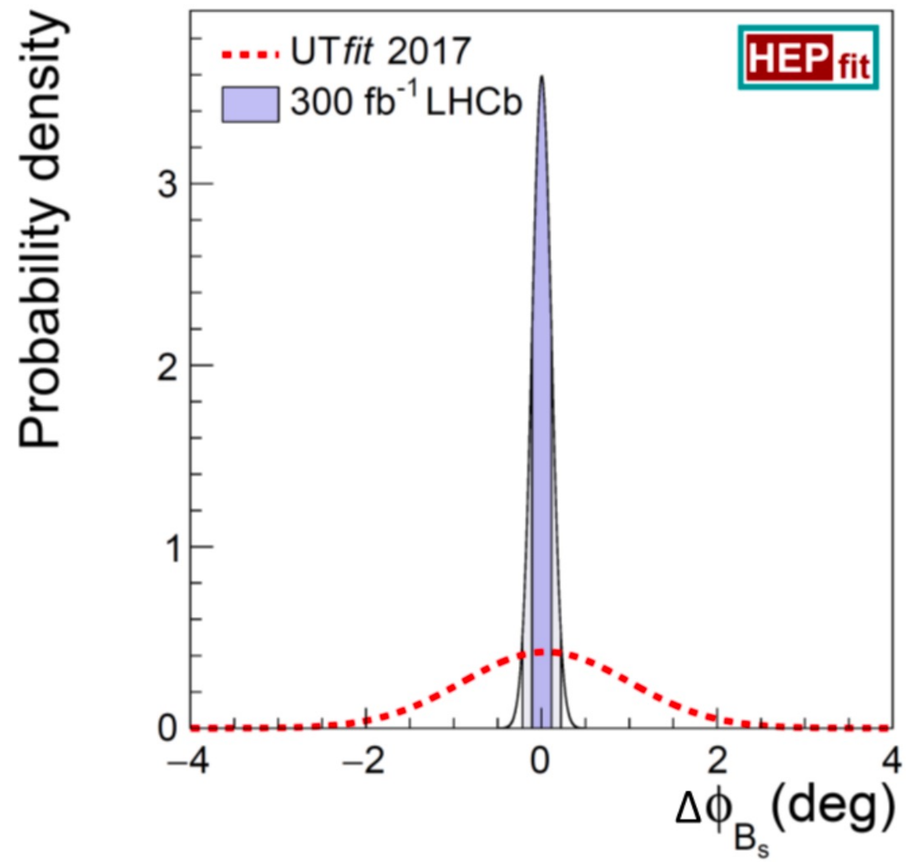
CKM triangle now



LHCb in the HL-LHC phase

- In case the LHCb upgrade in 2030 happens:





  - CKM triangle measurements will come on a new level