Particle Physics 2 : Exercise 9

1) Z decay rates

a) The cross section at $\sqrt{s} = m_Z$ is given in the lecture :

$$\sigma^0_{\text{ff}} = \frac{12\pi}{m_Z^2}\frac{\Gamma_{ee}\Gamma_{\text{ff}}}{\Gamma_Z^2},$$

which can be inverted to give

$$\Gamma_{ee}\Gamma_{\text{ff}} = \frac{\sigma^0_{\text{ff}} m_Z^2 \Gamma_Z^2}{12\pi}$$

Assuming lepton universality, whereby $\Gamma_{\mu\mu} = \Gamma_{\ell\ell}$

$$\Gamma_{ee}^2 = \frac{\sigma^0_{\mu\mu} m_Z^2 \Gamma_Z^2}{12\pi}$$

Converting the measured peak cross section of $\sigma^0(e^+e^- \to \ Z \to \ \mu^+\mu^-) = 1.9993$ nb into natural units gives

$$\begin{aligned}
\sigma^0(e^+e^- \to \ Z \to \ \mu^+\mu^-) &= 1.9993 \times 10^{-37}\ \text{m}^2 \cdot (\hbar c)^{-2} \\
&= 1.9993 \times 10^{-37}\ \text{m}^2 \frac{1}{(0.197\ \text{GeV} \times 10^{-15}\text{m})^2} \\
&= 5.152 \times 10^{-6}\ \text{GeV}^{-2}.
\end{aligned} \tag{1}$$

Using $m_Z = 91.1875$ GeV

$$\begin{aligned}
\Gamma_{ee}^2 &= \frac{\sigma^0_{\mu\mu} m_Z^2 \Gamma_Z^2}{12\pi} \\
&= \frac{5.152 \times 10^{-6} \cdot 91.1875^2}{12\pi}\Gamma_Z^2 \\
&- 1.136 \times 10^{-3}\Gamma_Z^2 \\
\Rightarrow \Gamma_{ee} &= 0.03371\Gamma_Z
\end{aligned} \tag{2}$$

Similarly,

$$\begin{aligned}
\Gamma_{ee}\Gamma_{\text{had}} &= \frac{\sigma^0_{\text{had}} m_Z^2 \Gamma_Z^2}{12\pi} \\
&= \frac{106.88 \times 10^{-6} \cdot 91.1875^2}{12\pi}\Gamma_Z^2 \\
&= 2.357 \times 10^2 \Gamma_Z^2 \\
\Rightarrow 0.0337\Gamma_Z\Gamma_{\text{hadrons}} &= 2.357 \times 10^2 \Gamma_Z^2 \\
\Rightarrow \Gamma_{\text{hadrons}} &= 0.6992\Gamma_Z
\end{aligned} \tag{3}$$

b) The total width of the Z is given by :

$$\Gamma_Z = 3\Gamma_{\ell\ell} + \Gamma_{\text{hadrons}} + N_\nu \Gamma_{\nu\nu}$$

From the results of part a) :

$$\begin{aligned}
N_\nu \Gamma_{\nu\nu} &= \Gamma_Z - 3\Gamma_{\ell\ell} - \Gamma_{\text{hadrons}} \\
&= 0.1997\Gamma_Z = 498\ \text{MeV}.
\end{aligned} \tag{4}$$

Updated: 25 avril 2024

Given the partial decay width for $Z \to \nu_e \nu_e$ is 167 MeV

$$N_\nu = \frac{498}{167} = 2.98,$$

consistent with the claim that there are three light neutrino generations.

2) Z cross section

The $e^+e^- \to Z \to \mathrm{f\bar{f}}$ differential cross section can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \kappa\Big[a(1+\cos^2\theta) + 2b\cos\theta\Big],$$

where a and b are constants related to the couplings to the Z, and κ is a normalisation factor. Writing $\cos\theta = x$, then $\mathrm{d}\Omega = 2\pi\mathrm{d}(\cos\theta) = 2\pi\mathrm{d}x$ and the number of events produced in the forward and backwards hemispheres can be written :

$$\begin{aligned} N_F &= 2\pi\kappa\int_0^1 a(1+x^2) + 2bx\mathrm{d}x \\ &= 2\pi\kappa\Big[\frac{4}{3}a + b\Big], \\ N_B &= 2\pi\kappa\int_{-1}^0 a(1+x^2) + 2bx\mathrm{d}x \\ &= 2\pi\kappa\Big[\frac{4}{3}a - b\Big]. \end{aligned} \tag{5}$$

Therefore the forward-backward asymmetry is

$$A_{\mathrm{FB}} = \frac{N_F - N_B}{N_F + N_B} = \frac{3b}{4a} \Rightarrow b = \frac{4aA_{\mathrm{FB}}}{3}.$$

Substituting this back into the original equation gives :

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \kappa\Big[a(1+\cos^2\theta) + \frac{8}{3}aA_{\mathrm{FB}}\cos\theta\Big], \\ &\propto (1+\cos^2\theta) + \frac{8}{3}A_{\mathrm{FB}}\cos\theta. \end{aligned}$$

3) Muon asymmetry

The muon asymmetry parameter is related to the couplings of the Z to muons by

$$\begin{aligned} \mathcal{A}_\mu = \frac{(c_L^\mu)^2 - (c_R^\mu)^2}{(c_L^\mu)^2 + (c_R^\mu)^2} &\equiv \frac{2c_V^\mu c_A^\mu}{(c_V^\mu)^2 + (c_A^\mu)^2} \\ &= \frac{2c_V^\mu/c_A^\mu}{(c_V^\mu/c_A^\mu)^2 + 1} \\ &= \frac{2x}{x^2+1} \end{aligned} \tag{6}$$

where $x = c_V^\mu/c_A^\mu$. Hence the measured value gives the quadratic equation

$$(0.1456 \pm 0.0091) = \frac{2x}{x^2+1}$$

$$\Rightarrow x^2 - (13.74 \pm 0.85)x + 1 = 0,$$

$$\Rightarrow x = 0.0732 \pm 0.0046 \quad \text{or} \quad x = 13.7 \pm 0.8.$$

In the Standard Model

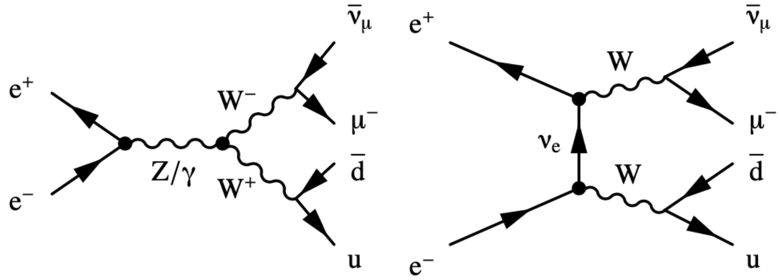
$$x = \frac{c_V^\mu}{c_A^\mu} = 1 - 4\sin^2\theta_W$$

and therefore the measurement can be interpreted as

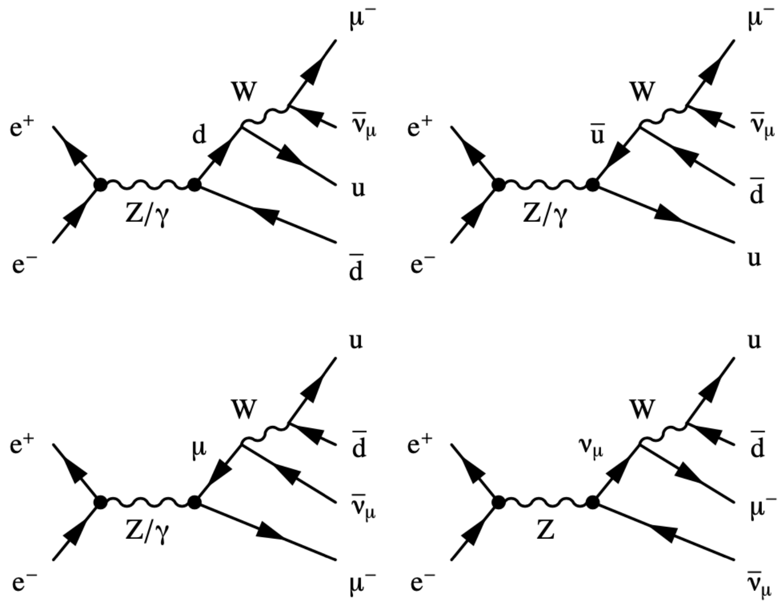
$$\begin{aligned} 1 - 4\sin^2\theta_W &= 0.0732 \pm 0.0046 \\ 4\sin^2\theta_W &= 0.9268 \pm 0.0046 \\ \sin^2\theta_W &= 0.2317 \pm 0.0012. \end{aligned} \tag{7}$$

4) **Feynman diagrams**

The first three diagrams (CC03) involve the production of two W bosons, either through the s-channel production of a Z or γ, or through the t-channel exchange of a neutrino. The remaining



seven diagrams, all arise from pair production of quarks or leptons through Z or γ exchange with a W radiated from one of the final state particles.



5) Top quark decay

a) In the rest frame of the top, the momenta of the two daughter particles are equal and conservation of energy implies $m_\mathrm{t} = E_\mathrm{b} + E_\mathrm{W}$. Writing this as $m_\mathrm{t} - E_\mathrm{b} = E_\mathrm{W}$ and squaring gives

$$\begin{aligned} m_\mathrm{t}^2 - 2m_\mathrm{t}E_\mathrm{b} + E_\mathrm{b}^2 &= E_\mathrm{W}^2 \\ m_\mathrm{t}^2 - 2m_\mathrm{t}E_\mathrm{b} + m_\mathrm{b}^2 + p^{*2} &= m_\mathrm{W}^2 + p^{*2} \\ \Rightarrow \quad m_\mathrm{t}^2 + (m_\mathrm{b}^2 - m_\mathrm{W}^2) &= 2m_\mathrm{t}E_\mathrm{b} \end{aligned} \tag{8}$$

Squaring again to eliminate E_b leads to

$$\begin{aligned} m_\mathrm{t}^4 + 2m_\mathrm{t}^2(m_\mathrm{b}^2 - m_\mathrm{W}^2) + (m_\mathrm{b}^2 - m_\mathrm{W}^2)^2 &= 4m_\mathrm{t}^2(m_\mathrm{b}^2 + p^{*2}) \\ m_\mathrm{t}^4 - 2m_\mathrm{t}^2(m_\mathrm{b}^2 + m_\mathrm{W}^2) + (m_\mathrm{W} - m_\mathrm{b})^2(m_\mathrm{W} + m_\mathrm{b})^2 &= 4m_\mathrm{t}^2p^{*2} \\ m_\mathrm{t}^4 - m_\mathrm{t}^2\Big[(m_\mathrm{W} + m_\mathrm{b})^2 + (m_\mathrm{W} - m_\mathrm{b})^2\Big] + (m_\mathrm{W} - m_\mathrm{b})^2(m_\mathrm{W} + m_\mathrm{b})^2 &= 4m_\mathrm{t}^2p^{*2} \\ \Big[m_\mathrm{t}^2 - (m_\mathrm{W} + m_\mathrm{b})^2\Big]\Big[m_\mathrm{t}^2 - (m_\mathrm{W} - m_\mathrm{b})^2\Big] &= 4m_\mathrm{t}^2p^{*2}, \end{aligned} \tag{9}$$

Thus showing that

$$p^* = \frac{1}{2m_\mathrm{t}}\sqrt{\Big[m_\mathrm{t}^2 - (m_\mathrm{W} + m_\mathrm{b})^2\Big]\Big[m_\mathrm{t}^2 - (m_\mathrm{W} - m_\mathrm{b})^2\Big]}.$$

Since $m_\mathrm{b} \ll m_\mathrm{W}$ the term inside the square root can be approximated to

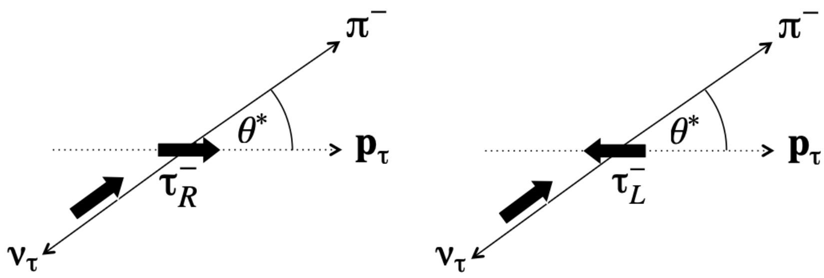
$$\begin{aligned} \Big[m_\mathrm{t}^2 - (m_\mathrm{W} + m_\mathrm{b})^2\Big]\Big[m_\mathrm{t}^2 - (m_\mathrm{W} - m_\mathrm{b})^2\Big] &= \left[m_\mathrm{t}^2 - m_\mathrm{W}^2\left(1 + \frac{m_\mathrm{b}}{m_\mathrm{W}}\right)^2\right]\left[m_\mathrm{t}^2 - m_\mathrm{W}^2\left(1 - \frac{m_\mathrm{b}}{m_\mathrm{W}}\right)^2\right] \\ &\approx \Big[m_\mathrm{t}^2 - m_\mathrm{W}^2 - 2m_\mathrm{W}m_\mathrm{b}\Big]\Big[m_\mathrm{t}^2 - m_\mathrm{W}^2 + 2m_\mathrm{W}m_\mathrm{b}\Big] \\ &= \Big[m_\mathrm{t}^2 - m_\mathrm{W}^2\Big]^2 - 4m_\mathrm{W}^2m_\mathrm{b}^2 \\ &\approx \Big[m_\mathrm{t}^2 - m_\mathrm{W}^2\Big]^2. \end{aligned} \tag{10}$$

Hence to a good approximation

$$p^* \approx \frac{m_\mathrm{t}^2 - m_\mathrm{W}^2}{2m_\mathrm{t}}$$

The same result could have been obtained much more quickly by simply neglecting the b-quark mass.

6) τ polarisation



a) Consider the decay in the tau rest frame with the angle θ^* defined with respect to the spin of the τ^-, as shown in the left-hand plot above. Since the neutrino will be left-handed, conservation of angular moment (just spin-half here) implies

$$\frac{\mathrm{d}N}{\mathrm{d}\cos\theta^*} \propto \cos^2\frac{\theta^*}{2} \propto 1+\cos\theta^*.$$

b) Consider the decay of a RH-helicity τ^- as shown in the left-hand plot above. With

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta^*} \propto \frac{(p^*)^2}{m_\tau}(1+\cos\theta^*).$$

In the laboratory frame the system will be boosted along the direction of the τ^- momentum and the laboratory-frame energy of the π^- will be

$$\begin{aligned} E_\pi &= \gamma_\tau E^* + \gamma_\tau\beta_\tau p_z^* \\ &= \gamma_\tau E^* + \gamma_\tau\beta_\tau p^* \cos\theta^*. \end{aligned} \tag{11}$$

The energy distribution of the pion in the laboratory frame is related to the angular distribution in the tau rest frame by

$$\begin{aligned} \frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} &= \frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta^*}\frac{\mathrm{d}\cos\theta^*}{\mathrm{d}E_\pi} \\ &\propto \frac{(p^*)^2}{m_\tau}(1+\cos\theta^*)\cdot\frac{1}{\gamma_\tau\beta_\tau p^*} \\ &\propto \frac{(p^*)^2}{\beta_\tau E_\tau}(1+\cos\theta^*). \end{aligned} \tag{12}$$

This can be expressed in terms of the pion energy using

$$\begin{aligned} E_\pi &= \gamma_\tau E^* + \gamma_\tau\beta_\tau p^*\cos\theta^*, \\ \Rightarrow \cos\theta^* &= \frac{E_\pi - \gamma_\tau E^*}{\gamma_\tau\beta_\tau p^*}, \end{aligned} \tag{13}$$

thus

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} \propto \frac{1}{\beta_\tau^2\gamma_\tau E_\tau}(E_\pi + \gamma_\tau\beta_\tau p^* - \gamma_\tau E^*). \tag{14}$$

In the laboratory frame the energy of the τ^- is just $m_\mathrm{Z}/2$ and therefore $\gamma_\tau = m_\mathrm{Z}/2m_\tau = 25.7$ and $\beta_\tau = 0.9992$. The momentum of the pion in the tau rest frame is easily shown to be

$$\begin{aligned} p^* &= \frac{m_\tau^2 - m_\pi^2}{2m_\tau} = 0.88 \text{ GeV} \approx m_\tau/2 \\ &\Rightarrow E^* = 0.89 \text{ GeV} \approx m_\tau/2. \end{aligned} \tag{15}$$

Consequently $\beta_\tau p^* - E^* < 0.02$ GeV and to a good approximation (Eq. 14) can be approximated as

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} &\propto \frac{1}{\beta_\tau^2\gamma_\tau E_\tau}(E_\pi + \gamma_\tau\beta_\tau p^* - \gamma_\tau E^*)\\ &\approx \frac{1}{\beta_\tau^2\gamma_\tau}\frac{E_\pi}{E_\tau}\end{aligned} \tag{16}$$

c) The case of the decay of a LH-helicity τ^- (as shown on the right in the above plot), the decay distribution in the tau rest frame can be obtained by replacing θ^* in the original decay distribution by $\pi - \theta^*$

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} &\propto \frac{(p^*)^2}{m_\tau}(1 + \cos[\pi - \theta^*])\\ &= \frac{(p^*)^2}{m_\tau}(1 - \cos\theta^*).\end{aligned} \tag{17}$$

Following the previous calculation the energy distribution of the decay pion will be

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} &= \frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta^*}\frac{\mathrm{d}\cos\theta^*}{\mathrm{d}E_\pi}\\ &\propto \frac{(p^*)^2}{\beta_\tau E_\tau}(1 - \cos\theta^*).\end{aligned} \tag{18}$$

where, as before, $\cos\theta^*$ is given by

$$\begin{aligned}E_\pi &= \gamma_\tau E^* + \gamma_\tau\beta_\tau p^*\cos\theta^*,\\ \Rightarrow \cos\theta^* &= \frac{E_\pi - \gamma_\tau E^*}{\gamma_\tau\beta_\tau p^*},\end{aligned} \tag{19}$$

and this

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\pi} &\propto \frac{1}{\beta_\tau^2\gamma_\tau E_\tau}(\gamma_\tau\beta_\tau p^* + \gamma_\tau E^* - E_\pi)\\ &\approx \frac{1}{\beta_\tau^2\gamma_\tau E_\tau}(\gamma_\tau m_\tau/2 + \gamma_\tau m_\tau/2 - E_\pi)\\ &\approx \frac{1}{\beta_\tau^2\gamma_\tau}\left(1 - \frac{E_\pi}{E_\tau}\right)\end{aligned} \tag{20}$$

where the following relations were used $\beta_\tau p^* \approx E^* \approx m_\tau/2$ and $\gamma_\tau m_\tau = E_\tau$.

d) From parts b) and c) the $\tau^- \to \pi^-\nu_\tau$ decays of RH and LH tau leptons give very different pion energy distributions, reflecting the different angular distributions of the decay relative to the tau line of flight :

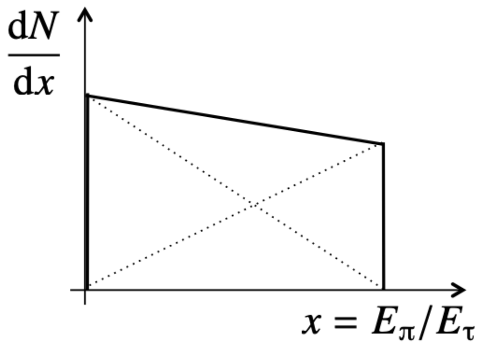
$$\frac{\mathrm{d}\Gamma_R}{\mathrm{d}E_\pi} \propto x \quad \text{and} \quad \frac{\mathrm{d}\Gamma_L}{\mathrm{d}E_\pi} \propto (1 - x).$$

where $x = E_\pi/E_\tau = 2E_\pi/m_\mathrm{Z}$. If the average τ^- polarisation is

$$P_\tau = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow},$$

and there are a total of $N = N_\uparrow + N_\downarrow$ decays, then

$$N_\uparrow = (1 + P_\tau)N/2 \quad \text{and} \quad N_\downarrow = (1 - P_\tau)N/2.$$



The corresponding pion energy distribution, which is correctly normalised, will be

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}x} &= 2N_{\uparrow}x + 2N_{\downarrow}(1-x) \\ &= N(1+P_\tau)x + N(1-P_\tau)(1-x) \\ &= N[(1-P_\tau) + 2P_\tau x]\end{aligned} \tag{21}$$

The observed distribution (shown schematically above) of

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}x} \propto 1.14 - 0.28x \equiv 0.86x + 1.14(1-x),$$

has contributions from LH and RH $\tau^- \to \pi^-\nu_\tau$ decays and implies that $P_\tau = -0.14$ and therefore (from the previous question)

$$\mathcal{A}_\tau = -P_\tau = 0.14.$$

The tau asymmetry parameter is related to the couplings of the Z to tau leptons by

$$\begin{aligned}\mathcal{A}_\tau = \frac{(c_L^\tau)^2 - (c_R^\tau)^2}{(c_L^\tau)^2 + (c_R^\tau)^2} &\equiv \frac{2c_V^\tau c_A^\tau}{(c_V^\tau)^2 + (c_A^\tau)^2} \\ &= \frac{2c_V^\tau/c_A^\tau}{(c_V^\tau/c_A^\tau)^2 + 1} \\ &= \frac{2x}{x^2+1}\end{aligned} \tag{22}$$

where $x = c_V^\tau/c_A^\tau$. Hence the measured value gives the quadratic equation

$$\begin{aligned}0.14 &= \frac{2x}{x^2+1} \\ \Rightarrow x^2 - 14.28x + 1 &= 0, \\ \Rightarrow x &= 0.067.\end{aligned} \tag{23}$$

In the Standard Model

$$x = \frac{c_V^\tau}{c_A^\tau} = 1 - 4\sin^2\theta_\mathrm{W}$$

and therefore the measurement can be interpreted as

$$\begin{aligned}1 - 4\sin^2\theta_\mathrm{W} &= 0.067 \\ 4\sin^2\theta_\mathrm{W} &= 0.933 \\ \sin^2\theta_\mathrm{W} &= 0.233.\end{aligned} \tag{24}$$