Particle Physics 2 : Exercise 2

1) Color in $e^+e^- \to$ q$\bar{\text{q}}$ measurements

To get the $e^+e^- \to \ \mu^+\mu^-$ cross section in nb we simply multiply it by $(\hbar c)^2$:

$$\sigma(\mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}(\hbar c)^2 \simeq \frac{4\pi}{3s} \times \left(\frac{0.197 \text{ GeV} \cdot \text{fm}}{137}\right)^2 \simeq 86.6 \text{ nb} \left(\frac{\text{GeV}^2}{s}\right)$$

Neglecting strong interaction effects, the cross section into hadrons can be estimated from the ratio

$$R = \frac{\sigma(e^+e^- \to \ \text{hadrons})}{\sigma(e^+e^- \to \ \mu^+\mu^-)} = C \sum_q Q_q^2$$

where C is the number of quark colors (3), Q_q^2 is the charge of the quark q (in units of e) and the sum includes those quarks for which $m(\text{q}\bar{\text{q}}) < \sqrt{s}$. At a centre-of-mass energy of 2 GeV, u, d and s quarks fulfill such condition, hence

$$\sigma(\text{hadrons}) = 3 \times \left(\frac{1}{9} + \frac{4}{9} + \frac{1}{9}\right) \times 86.6 \text{ nb} \left(\frac{\text{GeV}^2}{4 \text{ GeV}^2}\right) \simeq 43.3 \text{ nb}.$$

2) Hadron collider event kinematics

a) From the definition of rapidity

$$2y = \ln\left(\frac{E + p_z}{E - p_z}\right) \implies e^{2y} = \frac{E + p_z}{E - p_z},$$

which can be rearranged to give

$$E(e^{2y} - 1) = p_z(e^{2y} + 1) \implies \frac{E}{p_z} = \frac{e^{2y} + 1}{e^{2y} - 1} = \frac{e^y + e^{-y}}{e^y - e^{-y}} = \frac{\cosh y}{\sinh y}.$$

In the massless limit, the jet energy is the sum of the squares of the transverse and longitudinal momentum components $E^2 = p_\text{T}^2 + p_z^2$ and thus

$$\begin{aligned}
E^2 - p_z^2 &= p_\text{T}^2 \\
E^2\left(1 - \frac{p_z^2}{E^2}\right) &= p_\text{T}^2 \\
E^2\left(1 - \frac{\sinh^2 y}{\cosh^2 y}\right) &= p_\text{T}^2 \\
E^2 &= p_\text{T}^2 \cosh^2 y
\end{aligned}$$

from which it follows that $p_z^2 = E^2 - p_\text{T}^2 = p_\text{T}^2 \sinh^2 y$. Therefore the jet four-momentum can be written in terms of y, p_T and the azimuthal angle,

$$p = (p_\text{T} \cosh y,\ p_\text{T} \sin\phi, p_\text{T} \cos\phi,\ p_\text{T} \sinh y),$$

and since the two jets are here produced back-to-back in the transverse plane :

$$\begin{aligned}
p_3 &= (p_\text{T} \cosh y_3,\ p_\text{T} \sin\phi, p_\text{T} \cos\phi,\ p_\text{T} \sinh y_3) \\
p_4 &= (p_\text{T} \cosh y_4,\ p_\text{T} \sin\phi, p_\text{T} \cos\phi,\ p_\text{T} \sinh y_4).
\end{aligned}$$

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b) First recall that

$$\cosh y + \sinh y = \frac{1}{2}\left(e^{y} + e^{-y}\right) + \frac{1}{2}\left(e^{y} - e^{-y}\right) = e^{y} \tag{1}$$

$$\cosh y - \sinh y = \frac{1}{2}\left(e^{y} + e^{-y}\right) - \frac{1}{2}\left(e^{y} - e^{-y}\right) = e^{-y}. \tag{2}$$

Here conservation of energy and momentum implies

$$\frac{\sqrt{s}}{2}(x_1 + x_2) = p_{\mathrm{T}}(\cosh y_3 + \cosh y_4)$$
$$\text{and} \quad \frac{\sqrt{s}}{2}(x_1 - x_2) = p_{\mathrm{T}}(\sinh y_3 + \sinh y_4).$$

Taking the sum and the difference of these two relations and using the identities of (1) and (2) gives

$$\sqrt{s}x_1 = p_{\mathrm{T}}(\cosh y_3 + \cosh y_4 + \sinh y_3 + \sinh y_4) = p_{\mathrm{T}}\left(e^{y_3} + e^{y_4}\right)$$
$$\sqrt{s}x_2 = p_{\mathrm{T}}(\cosh y_3 + \cosh y_4 - \sinh y_3 - \sinh y_4) = p_{\mathrm{T}}\left(e^{-y_3} + e^{-y_4}\right),$$

and, as required,

$$x_1 = \frac{p_{\mathrm{T}}}{\sqrt{s}}\left(e^{y_3} + e^{y_4}\right) \quad \text{and} \quad x_2 = \frac{p_{\mathrm{T}}}{\sqrt{s}}\left(e^{-y_3} + e^{-y_4}\right).$$

c) In the massless limit ($p_1^2 = p_3^2 = 0$), the four-momentum transfer

$$\begin{aligned} Q^2 &= -q^2 = -(p_1 - p_3)^2 = 2p_1 \cdot p_3 \\ &= 2\frac{\sqrt{s}}{2}(x_1,\, 0,\, 0,\, x_1) \cdot (p_{\mathrm{T}}\cosh y_3,\; +p_{\mathrm{T}}\sin\phi, +p_{\mathrm{T}}\cos\phi, p_{\mathrm{T}}\sinh y_3) \\ &= p_{\mathrm{T}}\sqrt{s}x_1(\cosh y_3 - \sinh y_3) \\ &= p_{\mathrm{T}}\sqrt{s}x_1 e^{-y_3} \end{aligned}$$

Using the result for x_1 derived in part b) above,

$$Q^2 = p_{\mathrm{T}}\sqrt{s}x_1 e^{-y_3} = p_{\mathrm{T}}^2\left(e^{y_3} + e^{y_4}\right)e^{-y_3} = p_{\mathrm{T}}^2\left(1 + e^{y_4 - y_3}\right).$$

It is worth recalling that rapidity *differences* are invariant under boosts along the z direction, and therefore the result for (the Lorentz invariant) quantity Q^2 is invariant under boosts along the z direction (as it must be).

3) Drell-Yan cross section in pp and p$\bar{\text{p}}$ collisions

a) Writing $u(x) = u_V(x) + S(x)$ and $d(x) = d_V(x) + S(x)$,

$$\begin{aligned} \mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{p\bar{p}}} = &\frac{4\pi\alpha^2}{81 s x_1 x_2}\{4\left[u_V(x_1) + S(x_1)\right]\left[u_V(x_2) + S(x_2)\right] + 4S(x_1)S(x_2) + \\ &\left[d_V(x_1) + S(x_1)\right]\left[d_V(x_2) + S(x_2)\right] + S(x_1)S(x_2)\}\mathrm{d}x_1\mathrm{d}x_2 \\ = &\frac{4\pi\alpha^2}{81 s x_1 x_2}\{4u_V(x_1)u_V(x_2) + 4u_V(x_1)S(x_2) + 4S(x_1)u_V(x_2) + \\ &10S(x_1)S(x_2) + d_V(x_1)d_V(x_2) + d_V(x_1)S(x_2) + S(x_1)d_V(x_2)\}\mathrm{d}x_1.\mathrm{d}x_2 \end{aligned}$$

If we also assume that $u_V(x) = 2d_V(x)$ then

$$\begin{aligned} \mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{p\bar{p}}} = &\frac{4\pi\alpha^2}{81 s x_1 x_2}\{17d_V(x_1)d_V(x_2) + 9d_V(x_1)S(x_2) + \\ &9S(x_1)d_V(x_2) + 10S(x_1)S(x_2)\}\mathrm{d}x_1\mathrm{d}x_2. \end{aligned}$$

b) For pp collisions, the Drell-Yan cross section is

$$\mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{pp}} = \frac{4\pi\alpha^2}{81 s x_1 x_2}\{4u(x_1)\bar{u}(x_2) + 4\bar{u}(x_1)u(x_2) + \\ d(x_1)\bar{d}(x_2) + \bar{d}(x_1)d(x_2)\}\mathrm{d}x_1\mathrm{d}x_2.$$

This can be expressed in terms of sea and valence quark PDFs as

$$\mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{pp}} = \frac{4\pi\alpha^2}{81 s x_1 x_2}\{4u_V(x_1)S(x_2) + 4S(x_1)u_V(x_2) + \\ d_V(x_1)S(x_2) + S(x_1)d_V(x_2) + 10S(x_1)S(x_2)\}\mathrm{d}x_1\mathrm{d}x_2.$$

If we again assume that $u_V(x) = 2d_V(x)$ then

$$\mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{pp}} = \frac{4\pi\alpha^2}{81 s x_1 x_2}\{9d_V(x_1)S(x_2) + 9S(x_1)d_V(x_2) + 10S(x_1)S(x_2)\}\mathrm{d}x_1\mathrm{d}x_2.$$

c) Since $\hat{s} = x_1 x_2 s$, lines of constant $\hat{s}$ define hyperbolae in the $\{x_1, x_2\}$ plane. For $\hat{s} \ll s$ both x_1 and x_2 will usually be small. Therefore, in this region, the Drell-Yan cross section will be dominated by the sea quarks and, from the above results, $\mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{p\bar{p}}} \sim \mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{pp}}$. Consequently, the cross section for the Drell-Yan production of low-mass $\mu^+\mu^-$ pairs will be approximately the same for pp and p$\bar{\mathrm{p}}$ collisions. In contrast, for $\hat{s} > s/4$, both x_1 and x_2 will be (on average) greater than 0.5 and the valence quark contributions will dominate over the sea. In this case $\mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{p\bar{p}}} \gg \mathrm{d}^2\sigma_{\mathrm{DY}}^{\mathrm{pp}}$, and the cross section for the production of high-mass $\mu^+\mu^-$ pairs will be much greater for p$\bar{\mathrm{p}}$ collisions.