Particle Physics 2 : Exercise 11

1) Higgs potential

Introducing odd powers of the field ϕ into the Higgs potential would break the underlying gauge invariance of the Lagrangian, which is the whole point of introducing the Higgs mechanism in the first place.

2) Covariant derivative

This proof just requires care. The original Lagrangian is

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) = (\partial_\mu\phi^*\partial^\mu\phi) + ig(\partial_\mu\phi^*)B^\mu\phi - ig(\partial^\mu\phi)B_\mu\phi^* + g^2 B_\mu B^\mu\phi\phi^*.$$

Under the transformation

$$\phi(x) \to \ \phi'(x) = e^{ig\chi(x)}\phi(x) \quad \text{and} \quad B_\mu \to \ B'_\mu = B_\mu - \partial_\mu\chi(x),$$

this Lagrangian becomes

$$\begin{aligned}
\mathcal{L}' &= [\partial_\mu\phi^* - ig(\partial_\mu\chi)\phi^*][\partial^\mu\phi + ig(\partial^\mu\chi)\phi] \\
&\quad + ig[\partial_\mu\phi^* - ig(\partial_\mu\chi)\phi^*][B^\mu - \partial^\mu\chi]\phi \\
&\quad - ig[\partial^\mu\phi + ig(\partial^\mu\chi)\phi][B^\mu - \partial_\mu\chi]\phi^* \\
&\quad + g^2 B_\mu B^\mu\phi\phi^* - g^2[B_\mu(\partial^\mu\chi) + B^\mu(\partial_\mu\chi)]\phi\phi^* + g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi\phi^* \\
&= (\partial_\mu\phi^*)(\partial^\mu\phi) - ig(\partial_\mu\chi)(\partial^\mu\phi)\phi^* + ig(\partial_\mu\phi^*)(\partial^\mu\chi)\phi + g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi\phi^* \\
&\quad + ig(\partial_\mu\phi^*)B^\mu\phi + g^2(\partial_\mu\chi)B^\mu\phi^*\phi - ig(\partial_\mu\phi^*)(\partial^\mu\chi)\phi - g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi^*\phi \\
&\quad - ig(\partial^\mu\phi)B_\mu\phi^* + g^2(\partial^\mu\chi)B_\mu\phi\phi^* + ig(\partial^\mu\phi)(\partial_\mu\chi)\phi^* - g^2(\partial^\mu\chi)(\partial_\mu\chi)\phi\phi^* \\
&\quad + g^2 B_\mu B^\mu\phi\phi^* - g^2 B_\mu(\partial^\mu\chi)\phi\phi^* - g^2 B^\mu(\partial_\mu\chi)\phi\phi^* + g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi\phi^* \\
&= (\partial_\mu\phi^*)(\partial^\mu\phi) + ig(\partial_\mu\phi^*)B^\mu\phi - ig(\partial^\mu\phi)B_\mu\phi^* + g^2 B_\mu B^\mu\phi\phi^* \\
&= \mathcal{L}
\end{aligned} \tag{1}$$

3) Z and γ fields

The eigenvalues of the mass matrix are $\lambda_1 = 0$ and $\lambda_2 = g_\mathrm{W}^2 + g'^2$. The corresponding eigenvectors can be found by solving the equations

$$\mathbf{MX} = \begin{pmatrix} g_\mathrm{W}^2 & -g_\mathrm{W}g' \\ -g_\mathrm{W}g' & g'^2 \end{pmatrix}\mathbf{X} = \lambda\mathbf{X},$$

where $\mathbf{M}$ is the mass matrix giving mass terms of the form :

$$\begin{pmatrix} W_\mu^{(3)} & B_\mu \end{pmatrix}\mathbf{M}\begin{pmatrix} W_\mu^{(3)} \\ B_\mu \end{pmatrix}$$

For $\lambda = \lambda_1 = 0$ the eigenvalue equation gives

$$\begin{pmatrix} g_\mathrm{W}^2 & -g_\mathrm{W}g' \\ -g_\mathrm{W}g' & g'^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow g_\mathrm{W}^2 x_1 = g_\mathrm{W}g' x_2,$$

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and thus the corresponding eigenvector is :

$$\mathbf{X_1} = \frac{1}{\sqrt{g_{\mathrm{W}}^2 + g'^2}} \begin{pmatrix} g' \\ g_{\mathrm{W}} \end{pmatrix}$$

For the second eigenvalue,

$$\begin{pmatrix} g_{\mathrm{W}}^2 & -g_{\mathrm{W}} g' \\ -g_{\mathrm{W}} g' & g'^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (g_{\mathrm{W}}^2 + g'^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow g_{\mathrm{W}}^2 x_1 - g_{\mathrm{W}} g' x_2 = g_{\mathrm{W}}^2 x_1 + g'^2 x_1$$
$$\Rightarrow -g_{\mathrm{W}} x_2 = g' x_1,$$

and thus the corresponding eigenvalue is :

$$\mathbf{X_2} = \frac{1}{\sqrt{g_{\mathrm{W}}^2 + g'^2}} \begin{pmatrix} g_{\mathrm{W}} \\ -g' \end{pmatrix}$$

The mass eigenstates in the diagonal basis are therefore

$$\begin{pmatrix} A \\ Z \end{pmatrix} = \frac{1}{\sqrt{g_{\mathrm{W}}^2 + g'^2}} \begin{pmatrix} g' & g_{\mathrm{W}} \\ g_{\mathrm{W}} & -g' \end{pmatrix} \begin{pmatrix} W \\ B \end{pmatrix}$$

$$\Rightarrow A_\mu = \frac{g' W_\mu^{(3)} + g_{\mathrm{W}} B_\mu}{\sqrt{g_{\mathrm{W}}^2 + g'^2}} \quad \text{and} \quad Z_\mu = \frac{g_{\mathrm{W}} W_\mu^{(3)} - g' B_\mu}{\sqrt{g_{\mathrm{W}}^2 + g'^2}}.$$

4) HZZ coupling

The interaction terms in the Lagrangian arise from

$$\begin{aligned}
(D_\mu \phi)^\dagger (D^\mu \phi) &= \frac{1}{2}(\partial_\mu h \partial^\mu h) + \frac{1}{8} g_{\mathrm{W}}^2 (W_\mu^{(1)} + i W_\mu^{(2)})(W^{\mu(1)} - i W^{\mu(2)})(v+h)^2 \\
&+ \frac{1}{8}(g_{\mathrm{W}} W_\mu^{(3)} - g' B^\mu)(g_{\mathrm{W}} W^{\mu(3)} - g' B_\mu)(v+h)^2 \\
&= \frac{1}{2}(\partial_\mu h \partial^\mu h) + \frac{1}{8} g_{\mathrm{W}}^2 (W_\mu^{(1)} + i W_\mu^{(2)})(W^{\mu(1)} - i W^{\mu(2)})(v+h)^2 \\
&+ \frac{1}{8}(g_{\mathrm{W}}^2 + g'^2) Z^\mu Z_\mu (v+h)^2.
\end{aligned} \tag{2}$$

Considering the part of the expression involving the Z fields and using the relation $g' = g_{\mathrm{W}} \tan\theta_{\mathrm{W}}$

$$\begin{aligned}
\mathcal{L}_{\mathrm{Z}} &= \frac{1}{8}(g_{\mathrm{W}}^2 + g'^2) Z^\mu Z_\mu (v+h)^2 \\
&= \frac{g_{\mathrm{W}}^2}{8}(1 + \tan^2\theta_{\mathrm{W}}) Z^\mu Z_\mu (v+h)^2 \\
&= \frac{g_{\mathrm{W}}^2}{8\cos^2\theta_{\mathrm{W}}} Z^\mu Z_\mu (v+h)^2 \\
&= \frac{g_{\mathrm{W}}^2}{8\cos^2\theta_{\mathrm{W}}} (v^2 Z^\mu Z_\mu + 2vh Z^\mu Z_\mu + h^2 Z^\mu Z_\mu).
\end{aligned} \tag{3}$$

The first term in the brackets gives the Z mass :

$$
\begin{aligned}
&\frac{1}{2}m_{\mathrm{Z}}^2 Z^\mu Z_\mu = \frac{g_{\mathrm{W}}^2}{8\cos^2\theta_{\mathrm{W}}}v^2 Z^\mu Z_\mu \\
&\Rightarrow \quad m_{\mathrm{Z}} = \frac{1}{2}\frac{g_{\mathrm{W}}}{\cos\theta_{\mathrm{W}}}v = \frac{m_{\mathrm{W}}}{\cos\theta_{\mathrm{W}}.}
\end{aligned} \tag{4}
$$

The second term in the brackets of Eq. 3 gives the trilinear coupling between a physical Higgs field and two Z fields :

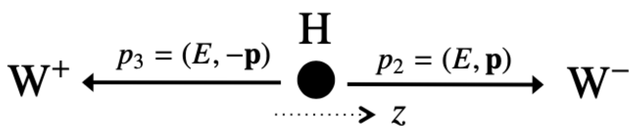
$$
\begin{aligned}
\mathcal{L}_{\mathrm{ZZh}} &= \frac{g_{\mathrm{W}}^2}{4\cos^2\theta_{\mathrm{W}}}vhZ^\mu Z_\mu \\
&= \frac{1}{2}\frac{g_{\mathrm{W}}}{cos\theta_{\mathrm{W}}}m_{\mathrm{Z}}hZ^\mu Z_\mu \\
&= \frac{1}{2}g_{\mathrm{Z}}m_{\mathrm{Z}}hZ^\mu Z_\mu .
\end{aligned} \tag{5}
$$

Hence the HZZ coupling is :

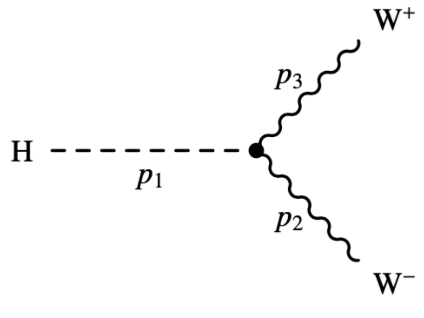
$$
g_{\mathrm{HZZ}} = \frac{1}{2}\frac{g_{\mathrm{W}}}{cos\theta_{\mathrm{W}}}m_{\mathrm{Z}}.
$$

5) H→ WW

a) and b) Consider the decay H → $\mathrm{W^+W^-}$



Denoting the respective four-momenta of the $\mathrm{W^-}$ and $\mathrm{W^+}$ as $p_2 = (E, 0, 0, p)$ and $p_3 = (E, 0, 0, -p)$. The corresponding Feynman diagram is shown below.



The relevant Feynman rules are a factor $ig_{\mathrm{W}}m_{\mathrm{W}}g_{\mu\nu}$ at the HWW vertex and factors of $\epsilon^\mu(p_2)^*$ and $\epsilon^\nu(p_3)^*$ for the final-state W bosons. The matrix element for the decay is

$$
-i\mathcal{M}_{fi} = ig_{\mathrm{W}}m_{\mathrm{W}}g_{\mu\nu}\epsilon^\mu(p_2)^*\epsilon^\nu(p_3)^*
$$

$$
\Rightarrow \mathcal{M}_{fi} = -g_{\mathrm{W}}m_{\mathrm{W}}g_{\mu\nu}\epsilon^\mu(p_2)^*\epsilon^\nu(p_3)^*.
$$

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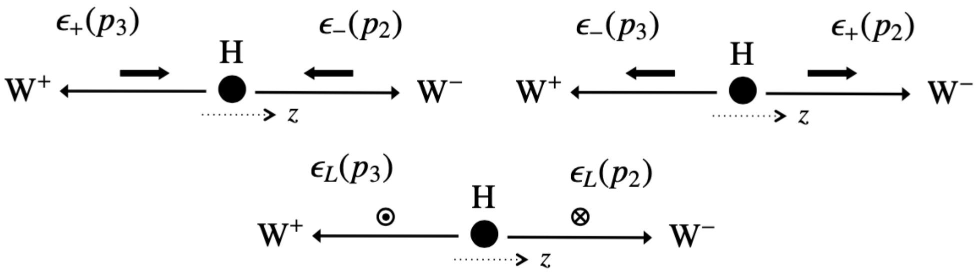
Thus the matrix element depends on the four-vector scalar product of the W boson polarisation four-vectors. The possible polarisation states are :

$$\epsilon_+^\mu(p_2)^* = \frac{1}{\sqrt{2}}(0,-1,i,0),\ \epsilon_-^\mu(p_2)^* = \frac{1}{\sqrt{2}}(0,1,i,0),\ \text{and}\ \epsilon_L^\mu(p_2)^* = \frac{1}{m_\mathrm{W}}(p,0,0,E),$$
$$\epsilon_+^\nu(p_3)^* = \frac{1}{\sqrt{2}}(0,-1,i,0),\ \epsilon_-^\nu(p_3)^* = \frac{1}{\sqrt{2}}(0,1,i,0),\ \text{and}\ \epsilon_L^\nu(p_3)^* = \frac{1}{m_\mathrm{W}}(-p,0,0,E),$$

where the $\pm$ refer to the gauge boson spin pointing in either the $\pm$ z-direction. Of the nine possible combinations, only three give non-zero four-vector scalar products :

$$\begin{aligned}\epsilon_+(p_2)^* \cdot \epsilon_-(p_3)^* &= +1,\\ \epsilon_-(p_2)^* \cdot \epsilon_+(p_3)^* &= +1,\\ \epsilon_L(p_2)^* \cdot \epsilon_L(p_3)^* &= \frac{-1}{m_\mathrm{W}^2}(p^2+E^2).\end{aligned}$$

The spin orientations for these three cases all correspond to spin-0 states as shown below.



The energy of each W is $E = m_\mathrm{H}/2$:

$$\begin{aligned}E^2 = \frac{m_\mathrm{H}^2}{4} &= p^2 + m_\mathrm{W}^2,\\ \Rightarrow p^2 &= \frac{1}{4}m_\mathrm{H}^2 - m_\mathrm{W}^2,\\ \Rightarrow E^2 + p^2 &= \frac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2,\end{aligned} \tag{6}$$

Therefore the three non-zero matrix elements are :

$$\begin{aligned}\mathcal{M}_{\downarrow\downarrow} &= -g_\mathrm{W} m_\mathrm{W},\\ \mathcal{M}_{\uparrow\uparrow} &= -g_\mathrm{W} m_\mathrm{W},\\ \mathcal{M}_{LL} &= +g_\mathrm{W}\frac{1}{m_\mathrm{W}}(p^2+E^2) = +g_\mathrm{W}\frac{1}{m_\mathrm{W}}(\frac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2,)\end{aligned}$$

where the arrows indicate the helicity states of the W bosons.

c) Since the Higgs boson is a spin-0 scalar, the spin-averaged matrix element squared is just :

$$\begin{aligned}\langle|\mathcal{M}|_{fi}^2\rangle &= \mathcal{M}_{\downarrow\downarrow}^2 + \mathcal{M}_{\uparrow\uparrow}^2 + \mathcal{M}_{LL}^2\\ &= 2g_\mathrm{W}^2 m_\mathrm{W}^2 + \frac{g_\mathrm{W}^2}{m_\mathrm{W}^2}\left(\frac{1}{2}m_\mathrm{H}^2 - m_\mathrm{W}^2\right)^2\\ &= g_\mathrm{W}^2 m_\mathrm{W}^2\left[2 + \left(\frac{m_\mathrm{H}^2}{2m_\mathrm{W}^2} - 1\right)^2\right].\end{aligned} \tag{7}$$

The total decay rate is therefore

$$
\begin{aligned}
\Gamma(\mathrm{H} \to \ \mathrm{W}^+\mathrm{W}^-) &= \frac{p}{8\pi m_\mathrm{H}^2}\langle|\mathcal{M}|_{fi}^2\rangle \\
&= \frac{g_\mathrm{W}^2 m_\mathrm{W}^2}{8\pi m_\mathrm{H}^2}\left(\frac{1}{4}m_\mathrm{H}^2 - m_\mathrm{W}^2\right)^{1/2}\left[2 + \left(\frac{m_\mathrm{H}^2}{2m_\mathrm{W}^2} - 1\right)^2\right] \\
&= \frac{g_\mathrm{W}^2 m_\mathrm{W}^2}{16\pi m_\mathrm{H}^2}\left(1 - \frac{4m_\mathrm{W}^2}{m_\mathrm{H}^2}\right)^{1/2}\left[\frac{m_\mathrm{H}^4}{4m_\mathrm{W}^4} - \frac{m_\mathrm{H}^2}{m_\mathrm{W}^2} + 3\right] \\
&= \frac{G_\mathrm{F} m_\mathrm{H}^3}{8\pi\sqrt{2}}\sqrt{1-4\lambda^2}(1 - 4\lambda^2 + 12\lambda^4),
\end{aligned} \tag{8}
$$

where $\lambda = m_\mathrm{W}/m_\mathrm{H}$.