Astrophysics IV   Observational Cosmology

Sheet 12: Assignments

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Exercise 1 : Fisher matrix

Fisher matrix analysis is widely used in astronomy. It measures the amount of information that an observable $\mathcal{O}$ carries about a parameter θ, which models $\mathcal{O}$. One important use of Fisher matrix in cosmology is to forecast the constraint on parameter θ given the model and uncertainties of the observable $\mathcal{O}$. Compared to a detailed end-to-end simulation, it provides a quicker and simpler way to predict the performance of an experiment before doing it, and can be used to guide the experimental design. Since the Fisher matrix forecast is purely analytical, it does not take into account practical errors, such as the receiver imperfections or residual foreground contamination that one might be able to introduce in a full end-to-end simulation. In addition, uncertainty in the observable $\mathcal{O}$ is treated as being Gaussian distributed, which might only be an approximation in practice. Therefore, the forecast from Fisher analysis gives the optimal results one could possibly expect from the upcoming experiment. One would not expect the results from a Fisher matrix forecast to be surpassed.

Formally, the Fisher matrix is a way to calculate the expected value of the observed information. Let $f(\mathcal{O};\theta)$ be the likelihood function for θ, which gives the probability distribution function of the observable $\mathcal{O}$ with an output o at a certain value of θ. The partial derivative $\frac{\partial}{\partial\theta}\log(\mathcal{O};\theta)$, of the natural logarithm of $f(\mathcal{O};\theta)$ with respect to θ, is known as the "score", which describes the sensitivity of $f(\mathcal{O};\theta)$ to the changes of θ. The expected value of the score is defined as

$$\mathrm{E}\left[\frac{\partial}{\partial\theta}\log f(\mathcal{O};\theta)\middle|\theta\right] = \int\left(\frac{\partial}{\partial\theta}\log f(o;\theta)\right)f(o;\theta)\mathrm{d}o\,. \tag{1}$$

The Fisher information $\mathcal{I}(\theta)$ is the variance of the score such that

$$\mathcal{I}(\theta) = \mathrm{E}\left[\left(\frac{\partial}{\partial\theta}\log f(\mathcal{O};\theta)\right)^2 \middle| \theta\right] = \int \left(\frac{\partial}{\partial\theta}\log f(o;\theta)\right)^2 f(o;\theta)\mathrm{d}o\,. \tag{2}$$

If $f(\mathcal{O};\theta)$ is twice differentiable with respect to θ, Equ. 2 can be written as

$$\mathcal{I}(\theta) = \int \left(\frac{\partial^2}{\partial^2\theta}\log f(o;\theta)\right) f(o;\theta)\mathrm{d}o\,. \tag{3}$$

In the case of forecasting cosmological parameters from CMB power spectrum measurement, the Fisher information can be written in a matrix format such that

$$M_{i,j} = \sum_{\ell=2}^{\ell_{\max}} \frac{\partial C_\ell}{\partial\theta_i}\,[\mathrm{Cov})]_\ell^{-1}\,\frac{\partial C_\ell}{\partial\theta_j} \tag{4}$$

where $\frac{\partial C_\ell}{\partial\theta_i}$ and $\frac{\partial C_\ell}{\partial\theta_j}$ are the derivative of the signal (power spectrum) w.r.t the two cosmological parameters. $[\mathrm{Cov}]_\ell$ is the covariance matrix of the measured power spectrum defined by

$$[\mathrm{Cov}]_\ell = \frac{2}{(2\ell+1)}\hat{C}_\ell^2, \tag{5}$$

where $\hat{C}_\ell$ is the measured power spectrum including both signal and noise. $[\mathrm{Cov}]_\ell$ quantifies the variance on the signal as a function of the angular scale (multipoles).

a) Calculate the noise power spectrum given an instrument setup for a CMB experiment
b) Compute the uncertainty on the measured CMB power spectrum
c) Estimate cosmological parameters given the measured power spectrum
d) Plot the confidence ellipse from the parameter estimation