Astrophysics V Observational Cosmology

Sheet 3: Solutions

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Exercise 1 : CMB radiation density

$T_0 = 2.73$ K, and thus $\rho_{\mathrm{CMB},0}\, c^2 = aT_0^4 \approx 4.2 \cdot 10^{-14}$ J m^{-3} and

$$\rho_{\mathrm{CMB},0} = 4.7 \cdot 10^{-31} kgm^{-3},$$

$$\Omega_{\mathrm{CMB},0} = \frac{8\pi G \rho_{\mathrm{CMB},0}}{3H_0^2} \approx 2.5h^{-2} \cdot 10^{-5}$$

The average photon energy is $3kT = 1.13 \cdot 10^{-22}$ J, corresponding to

$$n = 3.72 \cdot 10^8 \text{photons per m}^3 \sim 400 \text{photons per cm}^3.$$

$\Omega_{b,0} \approx 0.022h^{-2} = \rho_{\mathrm{b},0}/\rho_{\mathrm{crit},0}$ corresponds to the baryon number density

$$n_{\mathrm{b},0} = \frac{\rho_{\mathrm{b},0}}{m_{\mathrm{proton}}} \approx 0.25 \text{baryons per m}^3 \approx 2.5 \cdot 10^{-7} \text{baryons per cm}^{-3}$$

. The baryon density in the universe is much sparser than normal things on the earth.

Exercise 2 : Redshifting Planck's law

In the exercise we want to show that, if the CMB in the early Universe with scale radius R_1 is described by black body spectrum $u_{\nu 1} = \frac{8\pi\nu_1^3}{c^3}\frac{1}{\exp(h\nu_1/kT_1)-1}$ then also for a later Universe with scale radius R_2 the CMB remains described by a black body $u_{\nu 2} = \frac{8\pi\nu_2^3}{c^3}\frac{1}{\exp(h\nu_2/kT_2)-1}$.

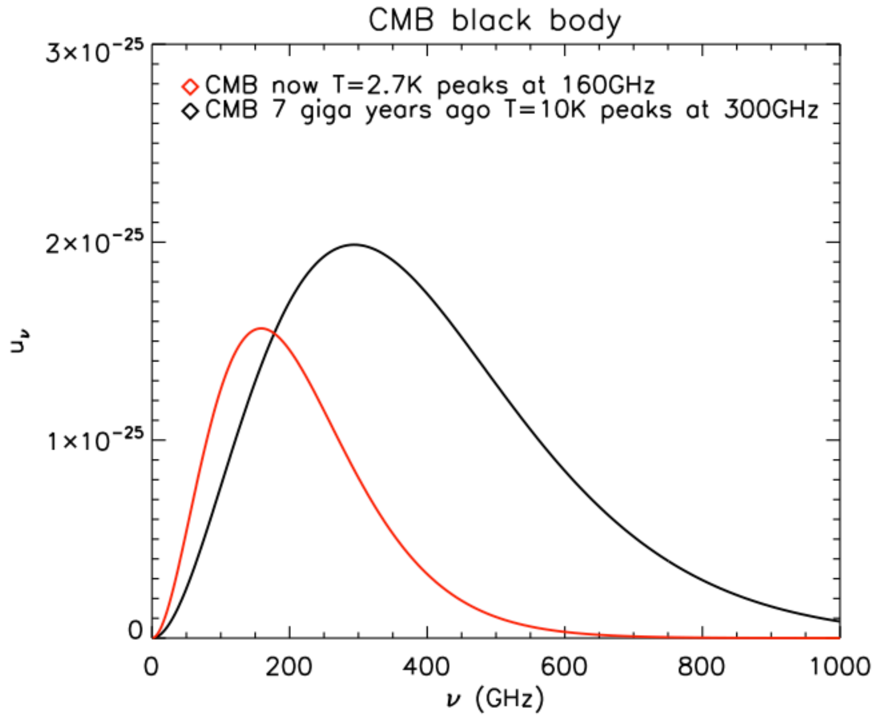


Figure 1. Despite the Universe's expansion CMB spectrum remains black body

a) The number density of photons (n) decreases in expanding Universe with increasing volume : $n_2 = (\frac{R_1}{R_2})^3 \cdot n_1$.
The energy of each photon ($h\nu$) also decreases with expanding Universe : $h\nu_2 = \frac{R_1}{R_2} h\nu_1$ (we have : $\lambda_0 = (1+z_i) \cdot \lambda_i$ and $R_i = (1+z_i)^{-1}$).
Hence, the total energy density of photons in Universe $U = n * h\nu$ is decreasing in expanding Universe : $U_2 = (\frac{R_1}{R_2})^4 \cdot U_1$

b) Knowing that, we take now our black body spectrum of the CMB at early Universe with scale radius R_1, and integrate it over whole spectral range :

$$U_2 = \left(\frac{R_1}{R_2}\right)^4 \cdot U_1 \rightarrow \int u_{\nu 2} d\nu_2 = \left(\frac{R_1}{R_2}\right)^4 \int u_{\nu 1} d\nu_1 \tag{1}$$

$$U_2 = \int u_{\nu 2} d\nu_2 = \left(\frac{R_1}{R_2}\right)^4 \int u_{\nu 1} d\nu_1 = \left(\frac{R_1}{R_2}\right)^4 \int \frac{8\pi\nu_1^3}{c^3} \frac{1}{\exp(h\nu_1/kT_1) - 1} d\nu_1$$

(2)

With using $h\nu_2 = \frac{R_1}{R_2} h\nu_1$:

$$U_2 = \int u_{\nu 2} d\nu_2 = \int \left(\frac{R_1}{R_2}\right)^4 \frac{8\pi\nu_2^3 \left(\frac{R_2}{R_1}\right)^3}{c^3} \frac{1}{\exp(h\nu_2 \frac{R_2}{R_1}/kT_1) - 1} d\nu_2 \frac{R_2}{R_1} \quad (3)$$

We recognise a black body spectrum with $T_2 = \frac{R_1}{R_2} T_1$: :

$$U_2 = \int u_{\nu 2} d\nu_2 = \int \frac{8\pi\nu_2^3}{c^3} \frac{1}{\exp(h\nu_2/kT_2) - 1} d\nu_2 \quad (4)$$

$$u_{\nu 2} = \frac{8\pi\nu_2^3}{c^3} \frac{1}{\exp(h\nu_2/kT_2) - 1} \quad (5)$$

Indeed for $t_2 > t_1$, the CMB is still black body.

c) As shown in Figure 2 A black-body spectrum with $T = 2.725$ K is the best-fit to the observed CMB spectrum. It is consistent with $2.7260 \pm 0.0013\,K$ mentioned in the lecture as 2.725K is within the 1σ error range.

d) The CMB emission at the time of emission and today has a similar shape, but the spectrum today is shifted to lower-energy (redshifted) due to the expansion of the Universe as shown in Figure 3.

Exercise 3 : Clustering of galaxies

In this exercise, we want to compute the estimator of the 2-point correlation function, starting from the number of pairs of galaxies separated by a distance r. Finally, we plot the resulting values and analyze them.

a) If n_{galaxies} is the total number of galaxies, the total number of pairs is :

$$N_{\text{allpairs}} = \frac{n_{\text{galaxies}} \times (n_{\text{galaxies}} - 1)}{2}. \quad (6)$$

Thus,

$$N_{\text{obs}}(r) = \frac{pairs(r)}{N_{\text{allpairs}}}, \quad (7)$$

where $pairs(r)$ is the number of pairs of galaxies separated by a distance r.

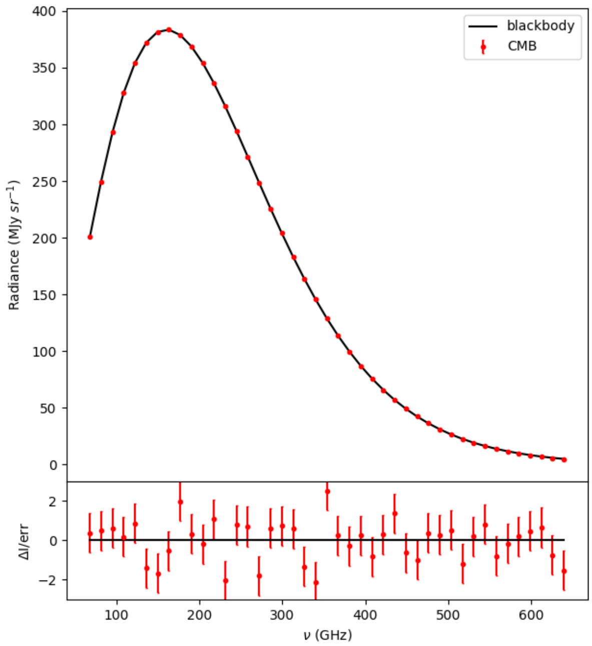


Figure 2. The CMB spectra at the time of emission and today.

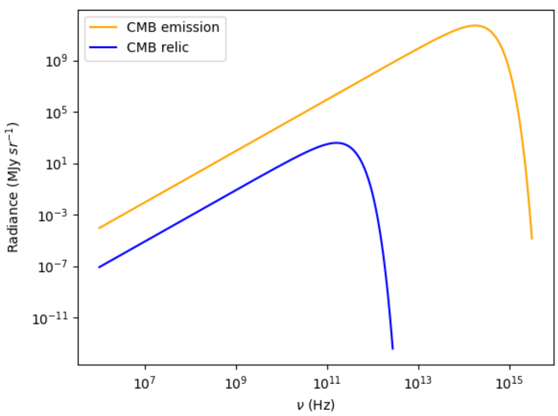


Figure 3. The best-fit black-body spectrum of CMB spectrum.

b) We assume that there is a point in the region dV_1 and we want to compute how many points are at a distance $r \pm \Delta r$, i.e. in the spherical shell of

radius r and width Δr. Given that the box has a finite size L, we need to assume the periodic boundary conditions. This is required mostly important for volumes $\mathrm{d}V_1$ that are close to the boundary, because the distance $r \pm \Delta r$ could be beyond the boundary. Thus, by applying the periodic boundary conditions, we ensure that there will be a complete spherical shell, even for those points that are close to the boundary.
Starting from the fact that N_{random} is defined as the ratio between the pairs of points separated by a distance $r \pm \Delta r$ over the total number of pairs in the catalog, we have :

$$N_{\text{random}} = \frac{\int_{\text{spherical shell}} \overline{n} \mathrm{d}V_2 \mathrm{d}V_1}{\int_{\text{whole box}} \overline{n} \mathrm{d}V_2 \mathrm{d}V_1}. \tag{8}$$

In this formula, $\overline{n}$ is the number density of random points and it is a constant because the points are uniformly distributed, thus it can be simplified. Moreover, the integration is done only over the V_2, thus $\mathrm{d}V_1$ can be also simplified, resulting into :

$$N_{\text{random}} = \frac{\int_{r-\Delta r}^{r+\Delta r} \int_0^{2\pi} \int_0^{\pi} s^2 \sin\theta \mathrm{d}s \mathrm{d}\theta \mathrm{d}\phi}{\int_0^L \int_0^L \int_0^L \mathrm{d}x \mathrm{d}y \mathrm{d}z}. \tag{9}$$

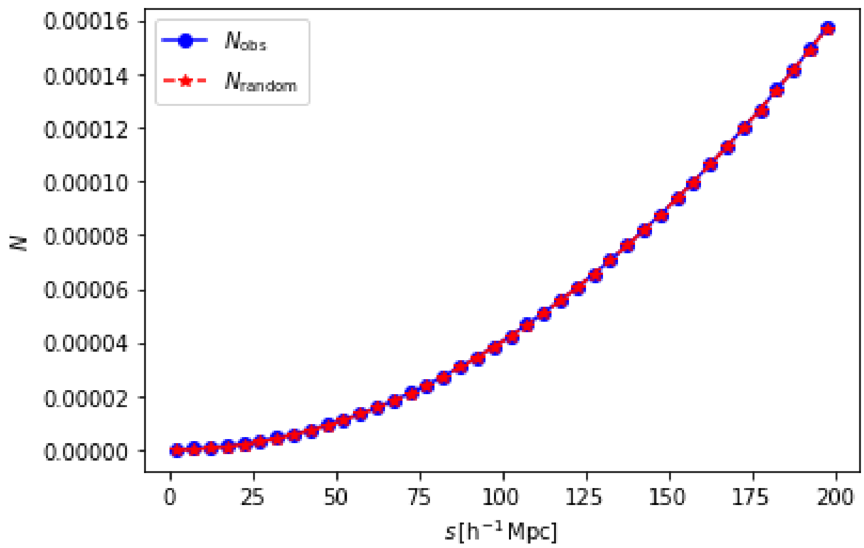
Finally,

$$N_{\text{random}}(r) = \frac{4\pi}{3} \times \frac{[(r+\Delta r)^3 - (r-\Delta r)^3]}{L^3}. \tag{10}$$

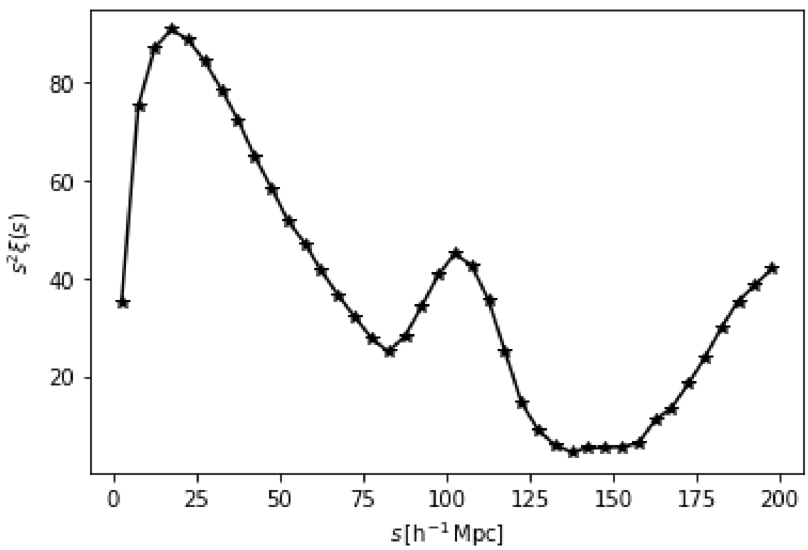
c) Using, the two results from above and the fact that

$$\xi(r) = \frac{N_{\text{obs}}(r)}{N_{\text{random}}(r)} - 1. \tag{11}$$

one can trivially compute $\xi(r)$ and plot it.



N_{random} and N_{obs} as functions of r



The 2PCF as function of r