Astrophysics V Observational Cosmology

Sheet 13: Solutions

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Exercise 1 : Friedmann Equations

a) From the second Friedmann equation :

$$\dot{R}^2 = \frac{8\pi \mathrm{G}}{3} R^2 \rho - k\,\mathrm{c}^2 + \frac{1}{3}\Lambda\,\mathrm{c}^2 R^2, \tag{1}$$

we can isolate the density term, and dividing out by R^2 we obtain :

$$\frac{8\pi \mathrm{G}}{3}\rho = \frac{\dot{R}^2}{R^2} + \frac{k\mathrm{c}^2}{R^2} - \frac{1}{3}\Lambda\,\mathrm{c}^2. \tag{2}$$

From the first Friedmann equation, we have :

$$\ddot{R} = -\frac{4\pi \mathrm{G}}{3}\left(\rho + \frac{3p}{\mathrm{c}^2}\right) R + \frac{1}{3}\Lambda\,\mathrm{c}^2 R, \tag{3}$$

Isolating the pressure term

$$\frac{4\pi \mathrm{G}}{\mathrm{c}^2} Rp = -\ddot{R} - \frac{4\pi \mathrm{G}}{3} R\rho + \frac{1}{3}\Lambda \mathrm{c}^2 R. \tag{4}$$

We can rewrite this :

$$\frac{8\pi \mathrm{G}}{\mathrm{c}^2} p = -2\,\frac{\ddot{R}}{R} - \frac{8\pi \mathrm{G}}{3}\rho + \frac{2}{3}\Lambda\,\mathrm{c}^2. \tag{5}$$

Substituting in this last equation the previous equation for the density :

$$\frac{8\pi \mathrm{G}}{\mathrm{c}^2} p = -2\,\frac{\ddot{R}}{R} - \left(\frac{\dot{R}^2}{R^2} + \frac{k\mathrm{c}^2}{R^2} - \frac{1}{3}\Lambda\,\mathrm{c}^2\right) + \frac{2}{3}\Lambda\,\mathrm{c}^2. \tag{6}$$

This can be re-written as :

$$\frac{8\pi \mathrm{G}}{\mathrm{c}^2} p = -2\,\frac{\ddot{R}}{R} - \frac{k\mathrm{c}^2}{R^2} - \frac{\dot{R}^2}{R^2} + \Lambda\,\mathrm{c}^2. \tag{7}$$

b) From equation 2, we can write :

$$8\pi \mathrm{G}\rho R^3 = 3k\mathrm{c}^2 R + 3R\dot{R}^2 - \Lambda \mathrm{c}^2 R^3. \tag{8}$$

Differentiating w.r.t time :

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\left(8\pi \mathrm{G}\rho R^3\right) &= 3k\mathrm{c}^2\dot{R} + 3\dot{R}^3 + 6R\dot{R}\ddot{R} - 3\Lambda\mathrm{c}^2 R^2\dot{R} \\ &= -3\dot{R}R^2\left(-\tfrac{k\mathrm{c}^2}{R^2} - \tfrac{\dot{R}^2}{R^2} - 2\tfrac{\ddot{R}}{R} + \Lambda\mathrm{c}^2\right). \end{aligned} \tag{9}$$

Substituting equation 7

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\mathrm{c}^2\, R^3\right) = -3\dot{R}R^2 p = -p\frac{\mathrm{d}R^3}{\mathrm{d}t}. \tag{10}$$

c) We remember that for matter the equation of state is $w = 0$ and for radiation $w = 1/3$. So for matter, we can write :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_M\mathrm{c}^2\, R^3\right) = 0 \qquad \Leftrightarrow \qquad \rho_M \propto R^{-3}. \tag{11}$$

For radiation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_r\mathrm{c}^2\, R^3\right) = -p_r\frac{\mathrm{d}R^3}{\mathrm{d}t} = -\frac{1}{3}\rho_r\mathrm{c}^2\frac{\mathrm{d}R^3}{\mathrm{d}t}, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_r\mathrm{c}^2\right) R^3 + \rho_r\mathrm{c}^2\frac{\mathrm{d}R^3}{\mathrm{d}t} = -\frac{1}{3}\rho_r\mathrm{c}^2\frac{\mathrm{d}R^3}{\mathrm{d}t}, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_r\mathrm{c}^2\right)\, R^3 = -\frac{4}{3}\rho_r\mathrm{c}^2\frac{\mathrm{d}R^3}{\mathrm{d}t} = -4\rho_r\mathrm{c}^2 R^2\dot{R}, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_r\mathrm{c}^2\right)\, R^4 = -4\rho_r\mathrm{c}^2 R^3\dot{R}. \tag{15}$$

This can be rewritten :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_r\mathrm{c}^2\right)\, R^4 + \rho_r\mathrm{c}^2\frac{\mathrm{d}R^4}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_r\mathrm{c}^2 R^4\right) = 0 \qquad \Leftrightarrow \qquad \rho_r \propto R^{-4}. \tag{16}$$

d) Stefan Boltzmann's law for a black body is $\rho_r = \frac{4\sigma}{\mathrm{c}}T^4$, so that the thermal law for radiation is :

$$R \cdot T = \text{constant}. \tag{17}$$