Astrophysics V   Observational Cosmology

Sheet 11: Solutions

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Exercise 1 : The BAO scale

For $z > z_{dec}$ the Universe was first dominated by radiation and then by matter ; the transition occurring at $z \sim 3000$. Thus the influence of dark energy can be supposed negligible : $\rho_\Lambda \ll \rho_\gamma$. Moreover matter is pressure-less : $p_b = 0$, and the expression of the total pressure is then :

$$p = p_b + p_\gamma + p_\Lambda \simeq p_\gamma = \frac{1}{3}\rho_\gamma c^2 \tag{1}$$

where in the last equality, we use the equation of state for photons.

With the same hypothesis, the total density ρ accounts only for baryonic matter and radiation :

$$\rho = \rho_b + \rho_\gamma \tag{2}$$

By replacing in the definition of the speed of sound, it leads to :

$$v_s^2 = \frac{c^2}{3}\frac{d\rho_\gamma}{d(\rho_b + \rho_\gamma)} = \frac{c^2}{3}\left[1 + \frac{d\rho_b}{d\rho_\gamma}\right]^{-1} \tag{3}$$

The baryon density scales with the scale factor like $\rho_b \propto a^{-3}$ and the radiation density like $\rho_\gamma \propto a^{-4}$. By differentiating the two one has that :

$$d\rho_b \propto \quad 3a^{-4}da \implies \quad \frac{d\rho_b}{\rho_b} = 3\frac{da}{a} \tag{4}$$

$$d\rho_\gamma \propto \quad 4a^{-5}da \implies \quad \frac{d\rho_\gamma}{\rho_\gamma} = 4\frac{da}{a} \tag{5}$$

Combining equations (4) and (5) leads to :

$$\frac{d\rho_b}{d\rho_\gamma} = \frac{3\rho_b}{4\rho_\gamma} \tag{6}$$

which, by replacing in equation (3), yields to the final expression :

$$v_s(z) = \frac{c}{\sqrt{3\left(1+\frac{3\rho_b(z)}{4\rho_\gamma(z)}\right)}} = \frac{c}{\sqrt{3\left(1+\frac{3\Omega_b^0}{4\Omega_\gamma^0(1+z)}\right)}} \tag{7}$$

where in the last equality we have used the fact that :

$$\frac{\rho_b(z)}{\rho_\gamma(z)} = \frac{\Omega_b(z)}{\Omega_\gamma(z)} = \frac{\Omega_b^0(1+z)^3}{\Omega_\gamma^0(1+z)^4} \tag{8}$$

for Ω_b^0 and Ω_γ^0 being the present day baryon and photon density parameters respectively. Notice that expression (7) can be approximated with a Taylor expansion, for $z > z_d$, into $v_s \simeq c/\sqrt{3}$.

Exercise 2 : CMB radiation density

$T_0 = 2.73$ K, and thus $\rho_{\text{CMB},0}\, c^2 = aT_0^4 \approx 4.2 \cdot 10^{-14}$ J m^{-3} and

$$\rho_{\text{CMB},0} = 4.7 \cdot 10^{-31} kgm^{-3},$$

$$\Omega_{\text{CMB},0} = \frac{8\pi G \rho_{\text{CMB},0}}{3H_0^2} \approx 2.5h^{-2} \cdot 10^{-5}$$

The average photon energy is $3kT = 1.13 \cdot 10^{-22}$ J, corresponding to

$$n = 3.72 \cdot 10^8 \text{photons per m}^3 \sim 400 \text{photons per cm}^3.$$

$\Omega_{b,0} \approx 0.022h^{-2} = \rho_{\text{b},0}/\rho_{\text{crit},0}$ corresponds to the baryon number density

$$n_{\text{b},0} = \frac{\rho_{\text{b},0}}{m_{\text{proton}}} \approx 0.25 \text{baryons per m}^3 \approx 2.5 \cdot 10^{-7} \text{baryons per cm}^{-3}$$

. The baryon density in the universe is much sparser than normal things on the earth.

Exercise 3 : Redshift of matter-radiation equality

From the definition of Ω_r and Ω_m we have :

$$\frac{\Omega_\text{r}}{\Omega_\text{m}} = \frac{\rho_\text{r}}{\rho_\text{m}} \tag{9}$$

The matter energy density scales as R^{-3}, that is :

$$\rho_\text{m} R^3 = \rho_{\text{m},0} R_0^3 \tag{10}$$

On the other hand, the photon/neutrino energy density scales as R^{-4}, as not only the volume expands as R^3, but the wavelength expands as well, and thus each photons energy shrinks as R^{-1} :

$$\rho_\text{r} R^4 = \rho_{\text{r},0} R_0^4 \tag{11}$$

Thus,

$$\frac{\Omega_{\rm r}}{\Omega_{\rm m}} = \frac{R_0}{R}\frac{\Omega_{\rm r,0}}{\Omega_{\rm m,0}} = (1+z)\frac{\Omega_{\rm r,0}}{\Omega_{\rm m,0}} \tag{12}$$

We want to find the redshift $z_{\rm eq}$ for which $\Omega_{\rm r}/\Omega_{\rm m} = 1$:

$$1 + z_{\rm eq} = \frac{\Omega_{\rm m,0}}{\Omega_{\rm r,0}} \tag{13}$$

Inserting the numerical value of $\Omega_{\rm r,0}$, this gives

$$1 + z_{\rm eq} \approx 23800\,\Omega_{\rm m,0}h^2 \approx 3500,$$

which corresponds to ~50,000 yr.

From lecture, we know that recombination occurred at $\sim 4000K$. We have $2.7(1 + z_{\rm rec}) = 4000$, thus $(1 + z_{\rm rec}) \approx 1400$. (A more careful treatment of the problem gives $(1 + z_{\rm rec}) \approx 1100$.)

Importantly, matter-radiation equality occurred before recombination.

Exercise 4 : Inflation

Flatness Problem : From the Friedmann equations, we see that the case of a universe with flat geometry is an unstable case. For the Universe to be flat today, i.e. $\Omega_k = 0$, then the value of Ω_k in the very early Universe, must have been exactly zero. It is the "fine tuning" of this scenario which is called the flatness problem.

Horizon Problem : The temperature fluctuations in the CMB, first measured by COBE (Bennet et al. 1990) were of the order $\Delta T/T \sim 2.10^{-5}$ at 10° angular scales on the sky. Yet the size of the causal horizon at the surface of last scattering (i.e. the distance travelled by a photon until the time of last scattering) on the sky correspond to ~ 1°. This last point means that areas of the sky separated by more than 1° could not have been in causal contact in the past - and so there is no reason for them to be the same temperature. This is in contradiction which what is observed in the CMB with COBE or WMAP data.

Inflation : Inflation was first brought forward by Guth & Tye (1980) and Guth (1981). The theory behind inflation suggests an early period of accelerated expansion, or inflation, after the Big Bang but before the surface of last scattering. This solved the Horizon problem since it means our whole observable Universe was inflated from a very small region at early times, so that all regions of the CMB we see today were all in causal contact in the early Universe. Inflation also solves the flatness problem. Whatever the initial curvature of the Universe was, it was smoothed out by the accelerated expansion. During inflation, quantum fluctuations in the early Universe are also expanded to astrophysical and cosmological scales (i.e. very large scales). These fluctuations are then thought to grow through gravitational instability and are therefore the *seeds of large scale structure* we see today.