Astrophysics V Observational Cosmology

Sheet 8: Solutions

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Exercise 1 : Newtonian lensing

a) To estimate the deflection angle of a photon lensed by a point mass, we want the ratio of its perpendicular velocity to its total velocity (once it reaches the observer). In the small angle approximation, we have :

$$\hat{\alpha} \approx \frac{v_\perp}{v_{\rm tot}} = \frac{v_\perp}{c} \tag{1}$$

To calculate $v_\perp$, we integrate the acceleration, assuming Newtonian gravity. From the Born approximation, we can assume that the photon path is straight, with impact parameter ξ. Because the lens is small compared to the total path length, we can take the limits of integration to be infinite.

$$v_\perp = \int_{-\infty}^{\infty} a_\perp dt = \int_{-\infty}^{\infty} \frac{GM}{(\xi^2 + z^2)} \frac{\xi}{(\xi^2 + z^2)^{1/2}} \frac{dz}{c} \tag{2}$$

Making the substitution $y = z/\xi$, we have :

$$v_\perp = \frac{GM}{c\,\xi} \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^{3/2}} = \frac{GM}{c\,\xi} \frac{y}{(1+y^2)^{1/2}} \Bigg|_{-\infty}^{\infty} = 2\frac{GM}{c\,\xi}, \tag{3}$$

where the integral can also be performed using trig substitution $y = \tan\theta$.

In total, we have :

$$\hat{\alpha} \approx \frac{2GM}{c^2\,\xi} \tag{4}$$

b) The Newtonian result is smaller by a factor of 2 than the result in General Relativity. This difference was one of the important early predictions of GR, and its confirmation provided strong evidence for the theory.

Exercise 2 : Gravitational lensing

a) From the figure, we can deduce, using the approximation of small angles :

$$\eta = \beta\, D_\mathrm{S} = \theta\, D_\mathrm{S} - \hat{\alpha}\, D_\mathrm{LS} \tag{5}$$

b) We use $\xi = \theta D_\mathrm{L}$, and combine the equations :

$$\beta = \theta - \frac{D_\mathrm{LS}}{D_\mathrm{S}} \frac{4GM}{\mathrm{c}^2\, \theta D_\mathrm{L}} = \theta - \frac{\theta_\mathrm{E}^2}{\theta} \tag{6}$$

and thus

$$\theta^2 - \beta\theta - \theta_\mathrm{E}^2 = 0 \tag{7}$$

which has the given solutions.

The negative solution θ_- is indeed a physical solution, it reprensents an angle measured in the opposite direction : as shown in Figure 7.7, the image is on the other side of the lens. Mathematically, this solution exists even if β is much larger than θ_E, that is if the source is located "somewhere else" on the sky. As we assume that the lens is a point mass, a particular light ray from the source can be as close as needed to the mass so that the deflection angle is high enough to redirect it towards the observer. For a lens having an extended mass distribution, this second image does not exist for a large β.

c) The distance between the images is

$$\theta_+ - \theta_- = 2 \cdot \frac{1}{2} \cdot \sqrt{\beta^2 + 4\theta_\mathrm{E}^2} \tag{8}$$

If β is small, this tends to $2\theta_\mathrm{E}$: the image separation is twice the Einstein radius.

Furthermore, in the Universe, a lens is never fully isolated. Any mass located in the vicinity of the line of sight will affect the lensing system.

d) The hint is to calculate the product of the image positions :

$$\theta_+ \cdot \theta_- = \frac{1}{4}\left(\beta^2 - (\beta^2 + 4\theta_\mathrm{E}^2)\right) = -\theta_\mathrm{E}^2 = -\frac{D_\mathrm{LS}}{D_\mathrm{L} D_\mathrm{S}} \frac{4GM}{\mathrm{c}^2} \tag{9}$$

and thus

$$M = |\theta_+ \theta_-| \cdot \frac{\mathrm{c}^2\, D_\mathrm{L} D_\mathrm{S}}{4G\, D_\mathrm{LS}} \tag{10}$$

Using

$$\begin{aligned} \mathrm{G} &= 6.673 \cdot 10^{-11}\ \mathrm{m^3\,s^{-2}\,kg^{-1}} & \quad 1\ M_\odot &= 1.989 \cdot 10^{30}\ \mathrm{kg} \\ \mathrm{c} &= 2.998 \cdot 10^{8}\ \mathrm{m\ s^{-1}} & \quad 1\ \mathrm{Mpc} &= 3.085 \cdot 10^{22}\ \mathrm{m} \end{aligned}$$

we obtain $M \simeq 8 \cdot 10^{11}\ M_\odot$.

Exercise 3 : Gravitational lenses - MACHOs

The mass of the MACHO is 10 Jupiter masses, $M = 1.899 \cdot 10^{28}$ kg. The distances under consideration are not cosmological, so we simply take $D_S = 50$ kpc, $D_L = 25$ kpc and $D_{LS} = 25$ kpc. Using the expression obtained in the previous exercise (Gravitational lensing) :

$$\theta_\pm = \frac{1}{2}\left(\theta_s \pm \sqrt{\theta_s^2 + 4\frac{4\mathrm{G}M}{\mathrm{c}^2}\frac{D_{LS}}{D_S D_L}}\right)$$

The Einstein radius θ_E is the radius of the ring that forms when the alignment between the source, the lens and the observer is perfect ($\theta_s = 0$), thus :

$$\theta_E = \sqrt{\frac{4\mathrm{G}M}{\mathrm{c}^2}\frac{D_{LS}}{D_S D_L}} = 1.91 \cdot 10^{-10} \text{ rad}$$

Since the speed of the MACHO is $v = 220$ km s $^{-1}$, this object will travel a distance of $2\theta_E D_L$ in time

$$t = 2\theta_E D_L/v = 1.34 \cdot 10^6 \text{ s} = 15.5 \text{ days}$$

This theoretical time gives us the order of magnitude of the duration of the magnification event. In the figure, it can be seen that the observed event lasted for approximately 20 days, which is in good agreement with our estimate.

Exercise 4 : Magnification of a galaxy luminosity function (Serjeant Ex. 7.11)

The background objects have the same redshift, so we could think of the luminosity function as differential source counts, $\mathrm{d}N/\mathrm{d}S \propto S^{-\alpha}$, for observed flux S. Therefore the number of objects per unit area brighter than a flux S_0 will be $N(> S_0) \propto S_0^{1-\alpha}$, which we could write as

$$N(> S_0) = k\, S_0^{1-\alpha} \tag{11}$$

Gravitational lensing conserves surface brightness : if the flux we get from a galaxy is increased by a factor μ due to the effect of the lens, the apparent angular size of this galaxy is also increased by the same factor μ. Hence, if the background galaxies are gravitationally magnified by a factor of μ, the intrinsic fluxes will be $S_{\text{intrinsic}} = S/\mu$, while the comoving volume sampled will be smaller by a factor of $1/\mu$. Said in other words, the sampled area of the sky will be μ times smaller, and the limiting intrinsic flux of the survey will be μ times lower. Therefore the number of galaxies brighter than an *observed* flux S_0 will be

$$N_{\text{lensed}}(> S_0) = \frac{k}{\mu}\left(\frac{S_0}{\mu}\right)^{1-\alpha} = k\,\mu^{-1}\, S_0^{1-\alpha}\,\mu^{\alpha-1} = k\, S_0^{1-\alpha}\,\mu^{\alpha-2} = N(> S_0)\cdot\mu^{\alpha-2} \tag{12}$$

Therefore for a magnification of μ (where $\mu > 1$), the lensing changes the number of background galaxies per unit area by a factor of $\mu^{\alpha-2}$. For this factor to be bigger than 1 we need

$$\log(\mu^{\alpha-2}) > \log(1) = 0 \quad \text{thus} \quad (\alpha - 2)\log(\mu) > 0. \tag{13}$$

We know that $\log(\mu) > 0$ for $\mu > 1$, and so α must be larger than 2. For example, if the source counts have a Euclidean slope ($\alpha = 2.5$), then lensing would increase the number of objects. The effects of sampling less volume due to lensing, and so finding fewer objects than the flux magnification on its own would suggest, is known as the *Broadhurst effect*.