Astrophysics V   Observational Cosmology

Sheet 6: Solutions

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Exercise 1 : Cosmological Constant vs. Dark Energy

In general cosmologists assume that the mysterious energy driving the Universe's acceleration is dark energy, i.e. a general quantity with general equation of state w, which can have a redshift dependence, i.e. $w = w(z)$. The dark energy equation of state is often parameterised as $w(z) = w_0 + (1 - a)w_a$.

Observational evidence suggests that the spatial distribution of dark energy is fairly smooth in the Universe, i.e. that there are little, if no, perturbations in the dark energy field (contrarily to the perturbations in the matter field, which are well observed). In practice, dark energy could clump at very large scales (scales beyond current observation).

The leading candidate for dark energy is the simplest one, where $w = -1$. In this case the equation of state is constant with redshift ($w_a = 0$ and $w_0 = -1,\ \forall z$). In the case of a cosmological constant, there are no perturbations in the 'dark energy' field, which can now be thought of as a 'vacuum energy' : a fixed amount of energy is attached to every tiny region of space, unchanging in time or spatial direction.

Exercise 2 : Critical Density

a) $\rho_c(z) = 3H^2(z)/8\pi \mathrm{G}$.

b) $\sim 10^{-27} kg\, m^{-3}$

c) ~ 10 Hydrogen atoms per m^3

d) $\sim 1.5 \times 10^{11} \mathrm{M}_\odot\, \mathrm{Mpc}^{-3}$.

Exercise 3 : Flatness problem

a) The critical density at z is $\rho_c = 3H^2/(8\pi G)$. With defining : $\rho = \rho_r + \rho_m$, $\Omega_r = \rho_r/\rho_c$, $\Omega_m = \rho_m/\rho_c$, $\Omega_k = -kc^2/(H^2a^2)$, and $\Omega_\Lambda = \Lambda c^2/(3H^2)$, we have :

$$1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda \tag{1}$$

b) With dividing the Friedmann equation by H_0^2, we obtain :

$$(H/H_0)^2 = \frac{\rho_r}{\rho_{c,0}} + \frac{\rho_m}{\rho_{c,0}} + \frac{\Omega_{k,0}}{a^2} + \Omega_{\Lambda,0} \tag{2}$$

As the volume universe scales with a, we have : $\rho_m = \rho_{m,0}/a^3$. Besides, as the photons lose energy via redshifting proportionnaly to $1/a$, we have : $\rho_r = \rho_{r,0}/a^4$. Thus :

$$(H/H_0)^2 = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{k,0}}{a^2} + \Omega_{\Lambda,0} \tag{3}$$

c) In a radiation-dominated Universe :

$$H^2(z) = (\dot{a}/a)^2 = \frac{\Omega_{r,0}}{a^4}. \tag{4}$$

Hence : $da/(a \times dt) \propto 1/a^2$, which means : $da/dt \propto 1/a$. We have. $a \propto t^{1/2}$ and $H \propto t^{-1}$.

In a matter-dominated Universe :

$$H^2(z) = (\dot{a}/a)^2 = \frac{\Omega_{m,0}}{a^3}. \tag{5}$$

We similarly obtain : $a \propto t^{2/3}$ and $H \propto t^{-1}$.

d) We have at t : $\Omega_k = -kc^2/(H^2a^2)$. Similarly at t_0 : $\Omega_{k,0} = -kc^2/(H_0^2)$ $(a_0 = 1)$. Hence :

$$|\Omega_k| \propto |\Omega_{k,0}|/(a^2H^2) \tag{6}$$

For a radiation-dominated Universe : $|\Omega_k| \propto t$. For a matter-dominated Universe : $|\Omega_k| \propto t^{2/3}$.

For the Universe to be flat today, i.e. $\Omega_{k,0} < 0.1$, given a matter-dominated approximation, we have the $|\Omega_k| \sim 10^{-13}$ back at the universe is 1 second years old. If Ω_k had been slightly above 0, it would have recollapsed very early before making galaxies ; If Ω_k had been slightly below 0, it would have expanded so rapidly that structures would not have formed. It is the "fine tuning" of this scenario which is called the flatness problem.

Exercise 4 : Redshift of matter-radiation equality

From the definition of Ω_r and Ω_m we have :

$$\frac{\Omega_\mathrm{r}}{\Omega_\mathrm{m}} = \frac{\rho_\mathrm{r}}{\rho_\mathrm{m}} \tag{7}$$

The matter energy density scales as R^{-3}, that is :

$$\rho_\mathrm{m} R^3 = \rho_{\mathrm{m},0} R_0^3 \tag{8}$$

On the other hand, the photon/neutrino energy density scales as R^{-4}, as not only the volume expands as R^3, but the wavelength expands as well, and thus each photons energy shrinks as R^{-1} :

$$\rho_{\rm r} R^4 = \rho_{\rm r,0} R_0^4 \tag{9}$$

Thus,

$$\frac{\Omega_{\rm r}}{\Omega_{\rm m}} = \frac{R_0}{R}\frac{\Omega_{\rm r,0}}{\Omega_{\rm m,0}} = (1+z)\frac{\Omega_{\rm r,0}}{\Omega_{\rm m,0}} \tag{10}$$

We want to find the redshift $z_{\rm eq}$ for which $\Omega_{\rm r}/\Omega_{\rm m} = 1$:

$$1 + z_{\rm eq} = \frac{\Omega_{\rm m,0}}{\Omega_{\rm r,0}} \tag{11}$$

Inserting the numerical value of $\Omega_{\rm r,0}$, this gives the desired result.