Astrophysics V   Observational Cosmology

Sheet 4: Solutions

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Exercise 1 : Scale factor and redshift (credits : B.Ryden)

a) The FLRW metric is

$$\begin{aligned} ds^2 &= -c^2dt^2 + a(t)^2 \cdot (dr^2 + S_\kappa(r)^2 \cdot d\Omega^2) \\ &= -c^2dt^2 + a(t)^2 \cdot (\frac{dx^2}{1-\kappa x^2} + x^2 \cdot d\Omega^2) \end{aligned}$$

Thus, we have

$$S_\kappa(r) = x, \tag{1}$$

and

$$dr^2 = \frac{dx^2}{1-\kappa x^2}. \tag{2}$$

Then

$$dr^2 = \begin{cases} \frac{dx^2}{1-x^2} \overset{x=sin\alpha}{=} \frac{(dsin\alpha)^2}{1-sin^2\alpha} = (d\alpha)^2 = (dsin^{-1}x)^2 & \text{for } \kappa = +1 \\ dx^2 & \text{for } \kappa = 0 \\ \frac{dx^2}{1+x^2} \overset{x=sinh\alpha}{=} \frac{(dsinh\alpha)^2}{1+sinh^2\alpha} = (d\alpha)^2 = (dsinh^{-1}x)^2 & \text{for } \kappa = -1. \end{cases} \tag{3}$$

Then the expression of $S_\kappa(r)$ is as follows

$$S_\kappa(r) = x = \begin{cases} sin(r) & \text{for } \kappa = +1 \\ r & \text{for } \kappa = 0 \\ sinh(r) & \text{for } \kappa = -1 \end{cases} \tag{4}$$

b) A photon follows a null geodesic : $ds = 0$. If the photon follows a radial path, then : $d\Omega^2 = 0$. We thus have : $dr = c \cdot dt/a(t)$. Integrating the previous equation, we have :

$$\int_{r'=0}^{r'=r} dr' = \int_{t'=t_e}^{t'=t_0} \frac{c \cdot dt'}{a(t')}.$$

Which gives :

$$r = c \int_{t_e}^{t_0} \frac{dt}{a(t)}.$$

c) The first wave crest is emitted at t_e and it is observed at t_0. The next wave crest is emitted at $t_e + \lambda_e/c$ and is observed at $t_0 + \lambda_0/c$. Consequently, we have :

$$r = c \int_{t_e}^{t_0} \frac{dt}{a(t)} = c \int_{t_e+\lambda_e/c}^{t_0+\lambda_0/c} \frac{dt}{a(t)}.$$

We subtract the $\int_{t_e+\lambda_e/c}^{t_0} dt/a(t)$ on both sides and divide by c :

$$\int_{t_e}^{t_e+\lambda_e/c} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\lambda_0/c} \frac{dt}{a(t)}$$

Besides, the expansion of the Universe does not change between two wave crests, as one can see for an optical photon :

$$\lambda/c \sim 2 \cdot 10^{-15}\ \mathrm{s} \ll \mathrm{H}_0^{-1} \sim 14\ \mathrm{Gyr}.$$

We can write :

$$\frac{1}{a(t_e)} \int_{t_e}^{t_e+\lambda_e/c} dt = \frac{1}{a(t_0)} \int_{t_0}^{t_0+\lambda_0/c} dt.$$

Hence we have :

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)}.$$

Using the redshift definition $z = (\lambda_0 - \lambda_e)/\lambda_e$ and the fact that $a(t_0) = 1$, we eventually have :

$$1 + z = \frac{1}{a(t_e)}.$$

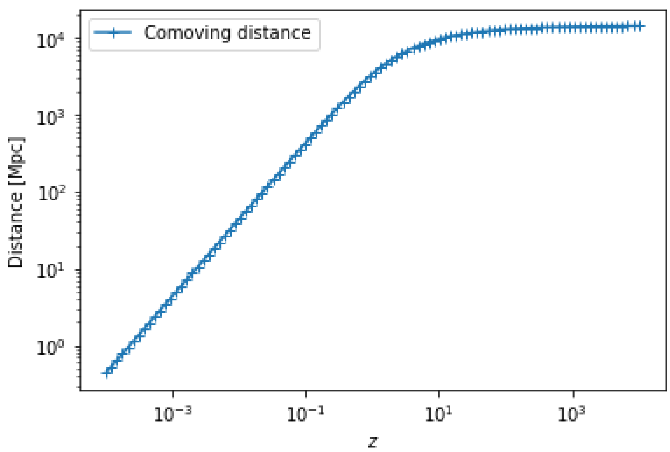

FIGURE 1 – **Figure 1.** Comoving distance as function of redshift

Exercise 2 : Luminosity distance (credits : B.Ryden)

a) $d_L \equiv \sqrt{\frac{L}{4\pi f}}$.

b) First, the wavelength of the light decreases from λ_e to λ_0 according to : $\lambda_0 = (1+z)\lambda_e$, which implies a decrease of the energy : $E_0 = E_e/(1+z)$.

Second, the time between two consecutive photon detection will be greater. If two photons are emitted in the same direction separated by a time interval δt_e, the proper distance between them is $c\delta t_e$; by the time we detect the photons at time t_0, the proper distance between them is stretched to $c\delta t_e \times (1+z)$. Thus, the two detected photons are separated in time by $\delta t_0 = \delta t_e \times (1+z)$.

Overall, the observed flux of light is decreased by a factor $(1+z)^2$.

c) The photons emitted at t_e are spread at t_0 over a sphere of proper radius $d_p(t_0)$, which is equal to the comoving radius r, as $d_p(t_0) = a(t_0) \cdot r$ and $a(t_0) = 1$. If the space is flat ($\kappa = 0$), the proper area of the sphere is given by the Euclidean relation $A_p(t_0) = 4\pi d_p(t_0)^2 = 4\pi r^2$.

d) Combining the fact that the observed flux is decreased by $(1+z)^2$ and that $A_p(t_0) = 4\pi S_\kappa(r)^2$, we obtain the relation :

$$f = \frac{L}{4\pi \cdot S_\kappa(r)^2 \cdot (1+z)^2}.$$

Hence we can express the luminosity distance as :

$$d_L = S_\kappa(r) \cdot (1+z).$$

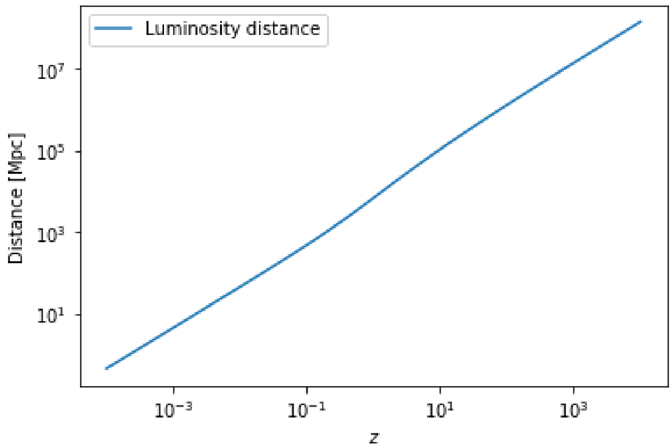

FIGURE 2 – **Figure 2.** The luminosity distance as a function of redshift

Exercise 3 : The evolution of surface brightness with redshift

a) The relation between the angular diameter distance and the luminosity distance is :

$$d_{\rm L} = (1+z)^2 d_{\rm A}$$

b) For a given proper size, the angular diameter will by definition be inversely proportional to $d_{\rm A}$. The angular area (for instance in square degrees) will thus vary as $\theta^2 \propto d_{\rm A}^{-2}$.

Using the definition of the luminosity distance, the flux will be inversely proportional to $d_{\rm L}^2$: $f \propto d_{\rm L}^{-2}$.

The surface brightness will therefore vary as $f/\theta^2 \propto d_{\rm A}^2/d_{\rm L}^2$. But $d_{\rm L} = (1+z)^2 d_{\rm A}$, so surface brightness must vary as $(1+z)^{-4}$.

Exercise 4 : Angular diameter distance

a) In the considered model $(\Omega_M, \Omega_\Lambda) = (1,0) \;\Rightarrow\; k=0$, we have :

$$d_{\rm A}(z) = \frac{d_{\rm comoving}}{1+z}$$

And thus :

$$\begin{aligned} d_{\rm A}(z) &= \frac{c}{H_0(1+z)} \int_0^z \frac{1}{\sqrt{(1+z')^2(1+z'\,\Omega_{\rm m,0}) - z'(2+z')\,\Omega_{\Lambda,0}}}\,{\rm d}z' \\ &= \frac{c}{H_0(1+z)} \int_0^z (1+z')^{-3/2}\,{\rm d}z' = \frac{c}{H_0(1+z)} \left[-2(1+z')^{-1/2}\right]_0^z \\ &= \frac{2c}{H_0}\,\frac{1+z-\sqrt{1+z}}{(1+z)^2} \end{aligned}$$

b) The angular diameter $\theta = D/d_{\rm A}$ reaches its minimum if $d_{\rm A}(z)$ is maximal :

$$\begin{aligned} \frac{\rm d}{{\rm d}z} d_{\rm A}(z) &= \frac{2c}{H_0}\left(\left(1+z-\sqrt{1+z}\right)\frac{\rm d}{{\rm d}z}\left[(1+z)^{-2}\right] + \frac{1}{(1+z)^2}\frac{\rm d}{{\rm d}z}\left[1+z-\sqrt{1+z}\right]\right) \\ &= \frac{2c}{H_0}\left(\frac{-2\left(1+z-\sqrt{1+z}\right)}{(1+z)^3} + \frac{1}{(1+z)^2}\left[1-\frac{1}{2}(1+z)^{-1/2}\right]\right) = 0 \\ \Leftrightarrow &\quad -2\left(1+z-\sqrt{1+z}\right) + (1+z) - \frac{1}{2}\sqrt{1+z} = 0 \\ \Leftrightarrow &\quad \frac{3}{2}\sqrt{1+z} = (1+z) \qquad \Leftrightarrow \qquad z = 1.25 \end{aligned}$$

Thus, for the Einstein-de Sitter Universe, the angular diameter distance increases with z for $z < 1.25$ then decreases for $z > 1.25$.

c)

$$d_{\rm A}(1.25) \approx 0.3\frac{c}{H_0}$$

$$\theta_{\rm min} = \frac{D}{d_{\rm A(1.25)}} \approx \frac{30\,{\rm kpc}}{0.3\frac{c}{H_0}} \approx 33\cdot 10^{-6}\,h\,{\rm rad} \approx 6.8\,h \text{ arc seconds.}$$

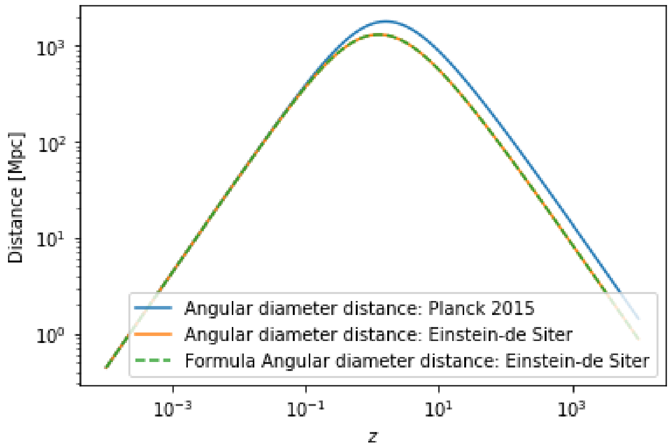

FIGURE 3 – **Figure 3.** The angular distance as a function of redshift z. Solid lines represent the results provided by astropy.cosmology package. The dashed line is the result provided by the analytical expression derived in this exercise.

d) The intrinsic luminosity is $L = 10^{43}$ erg s^{-1}, and the observed flux is $l = 3 \cdot 10^{-14}$ erg s^{-1} cm^{-2}. From this we can obtain the luminosity distance :

$$d_{\mathrm{L}} = \left(\frac{L}{4\pi l}\right)^{1/2} = 1670 \text{ Mpc}$$

For a redshift of $z = 0.5$, this implies an angular diameter distance of

$$d_{\mathrm{A}} = d_{\mathrm{L}}/(1+z)^2 = 740 \text{ Mpc}$$

Note that this corresponds to the evaluation of $d_{\mathrm{A}}(z)$ as obtained above for $H_0 = 100$ km s^{-1} Mpc^{-1}. But given the observations, we don't need to assume a value for H_0.

Finally, the proper diameter of the galaxy is given by :

$$D = d_{\mathrm{A}}\,\theta \approx 20 \text{ kpc}$$