

Stellar orbits

4th part

Outlines

Orbits in planar non-axisymmetric potential

- Surface of sections
 - energy dependency
 - flattening dependency
- Integrals of motions

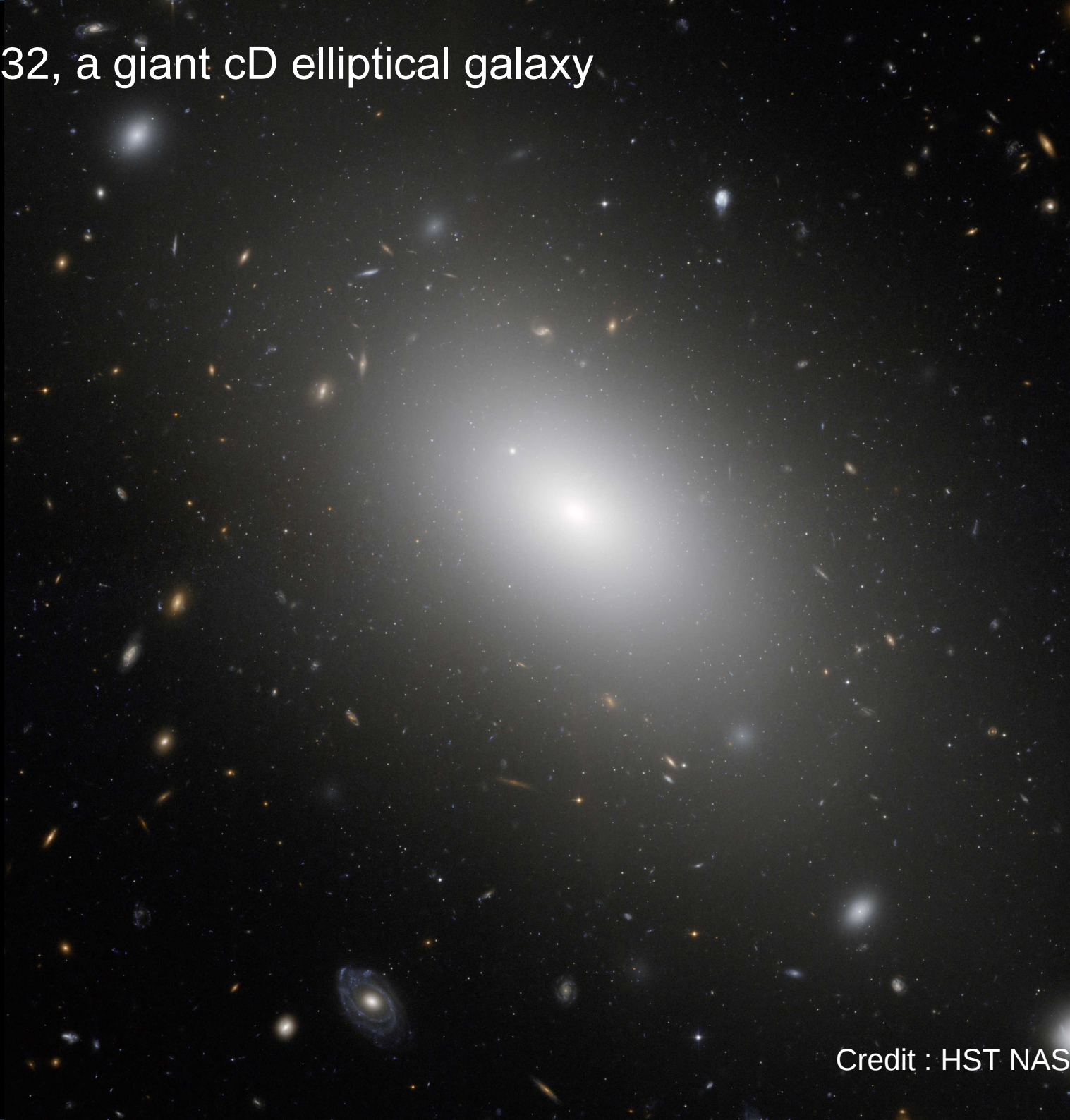
Orbits in planar non-axisymmetric rotating potential

- The Jacobi integral
- Lagrange points
- Orbits around Lagrange points
- Orbits not confined to Lagrange points

Stellar Orbits

Orbits in planar non-axisymmetric potentials

NGC 1132, a giant cD elliptical galaxy



Credit : HST NASA/ESA

NGC 1300 SBb



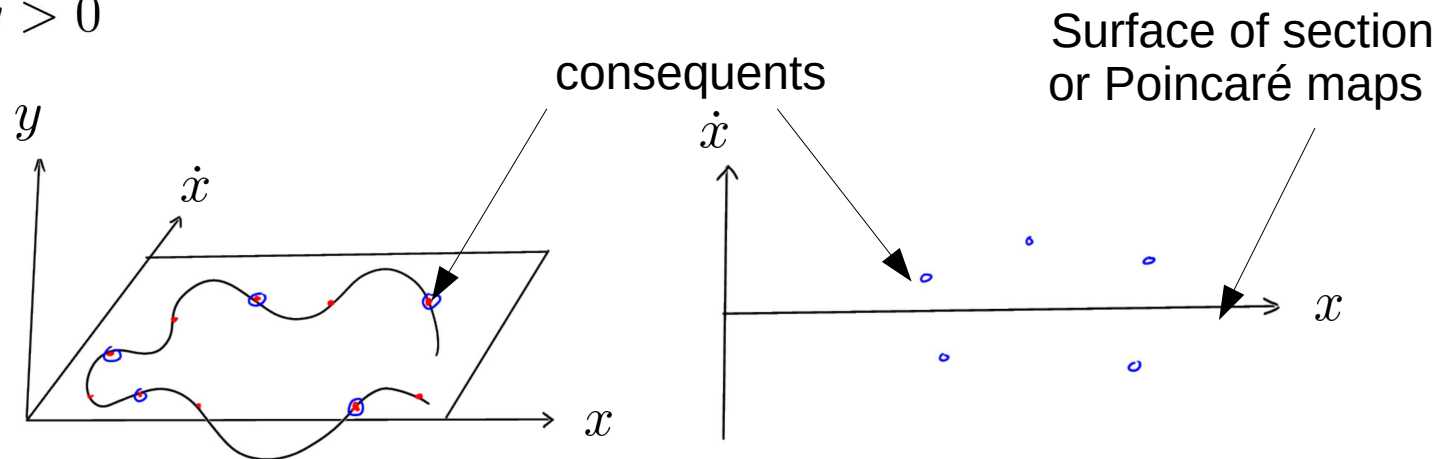
Surfaces of section (in planar potentials)

Can we visualize the phase phase and check if an additional integral of motion exists ?

Idea :

We study the orbits in the plane $z=0$

- 4-D 4 indep. variables $(x, y, \dot{x}, \dot{y})$
- Energy E
→ 3-D 3 indep. variables $(x, y, \dot{x})$
- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
 - cross the $y = 0$ plane
 - have $\dot{y} > 0$

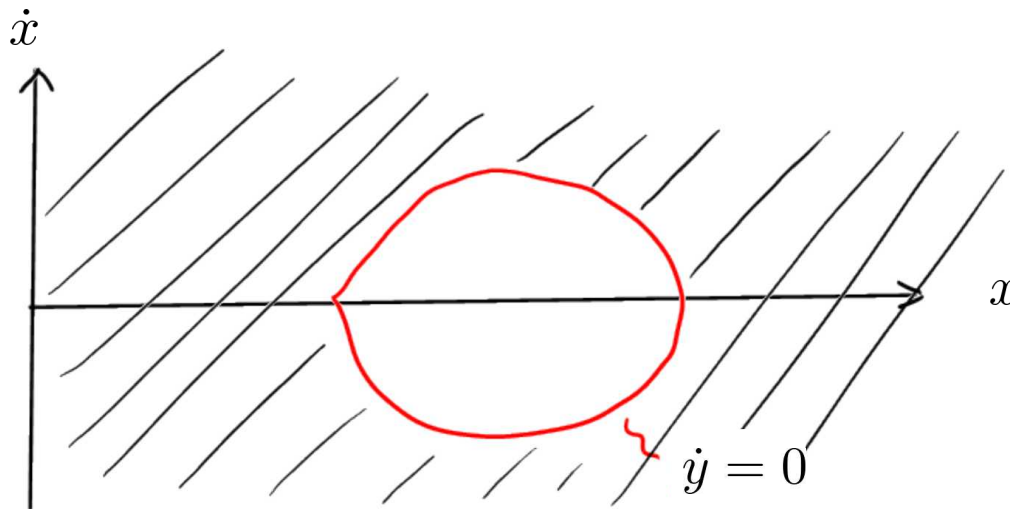


Surfaces of section (in planar potentials)

- A point in the surface of section (for a given E) defines an orbit as the three independent variables $(x, \dot{x}, y = 0)$ are defined.
- Even if orbits have the same energy, they will never intersect in the plane.
- Zero velocity curve : curve defined by $\dot{y} = 0$

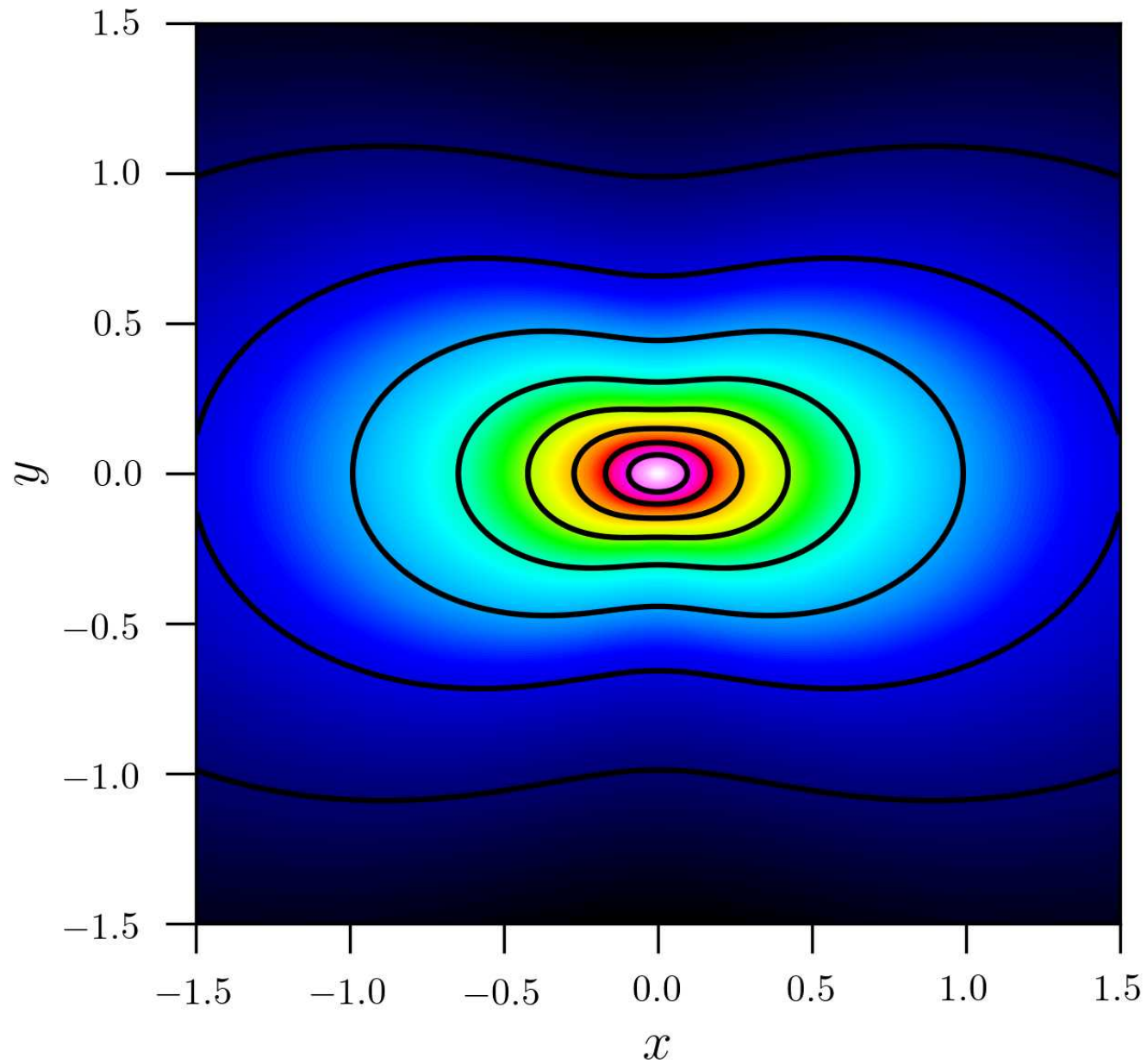
$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \Phi(x, y = 0) \quad \Rightarrow \quad \dot{x} \leq \pm \sqrt{2[E - \Phi(x, y = 0)]}$$

$\dot{x}(x) = \pm \sqrt{2[E - \Phi(x, y = 0)]}$ defines the accessible region of the phase space



Bar model : Logarithmic potential:
Vo=1 Rc=0.13 q=0.8)

$$\Phi_{\log}(x, y) = \frac{1}{2} V_0^2 \ln \left(R_c^2 + x^2 + \left(\frac{y}{q} \right)^2 \right)$$



$$R \ll R_c$$

Orbits in planar non-axisymmetric static potential

Model : logarithmic potential

$$\phi(x, y) = \frac{1}{2} V_0^2 \ln \left(R_c + x^2 + \frac{y^2}{q^2} \right)$$

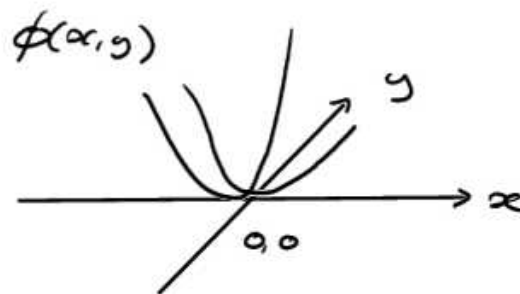
q : flattening parameter
(equipotential axis ratio)

Motions for $R \ll R_c$

$$\phi(x, y) \approx \phi(0, 0) + \overset{=0}{\cancel{\frac{\partial \phi}{\partial x} \Big|_{0,0}}} x + \overset{=0}{\cancel{\frac{\partial \phi}{\partial y} \Big|_{0,0}}} y + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_{0,0} x^2 + \frac{1}{2} \frac{\partial^2 \phi}{\partial y^2} \Big|_{0,0} y^2$$

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{0,0} = \frac{V_0^2}{R_c^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{0,0} = \frac{V_0^2}{R_c^2} \frac{1}{q^2}$$



Equations of motion

$$\ddot{x} = - \frac{\partial \phi}{\partial x}$$

$$\ddot{y} = - \frac{\partial \phi}{\partial y}$$

$\rightarrow$

$$\ddot{x} = - \frac{V_0^2}{R_c^2} x$$

$$\ddot{y} = - \frac{V_0^2}{q^2 R_c^2} y$$

$$\omega_x = \frac{V_0}{R_c}$$

$$\omega_y = \frac{V_0}{q R_c}$$

2 decoupled harmonic oscillators
with different frequencies

$$\omega_y = \frac{1}{q} \omega_x \quad (q < 1)$$

$$\text{if } q = \frac{n}{m} \quad n, m \in \mathbb{N}$$

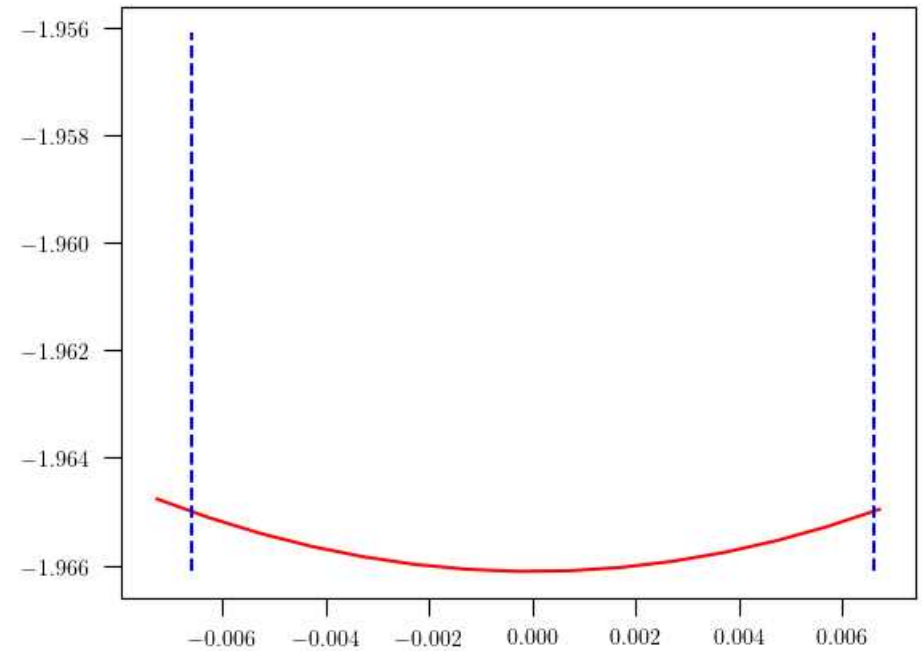
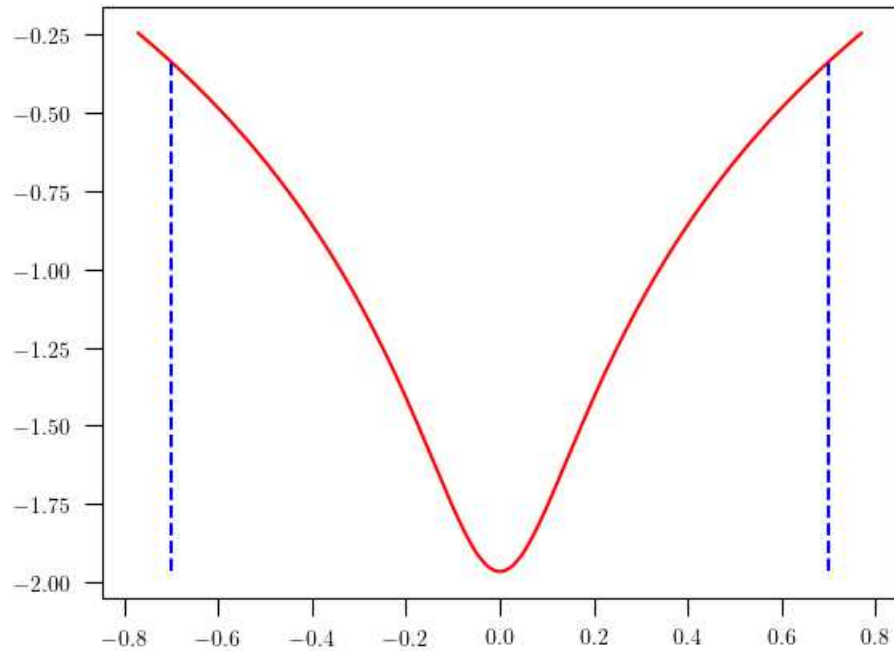
$\Rightarrow$ closed orbit

Integrals of motions (Hamiltonians)

$$H_x = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega_x^2 x^2$$

$$H_y = \frac{1}{2} \dot{y}^2 + \frac{1}{2} \omega_y^2 y^2$$

Potential and energy

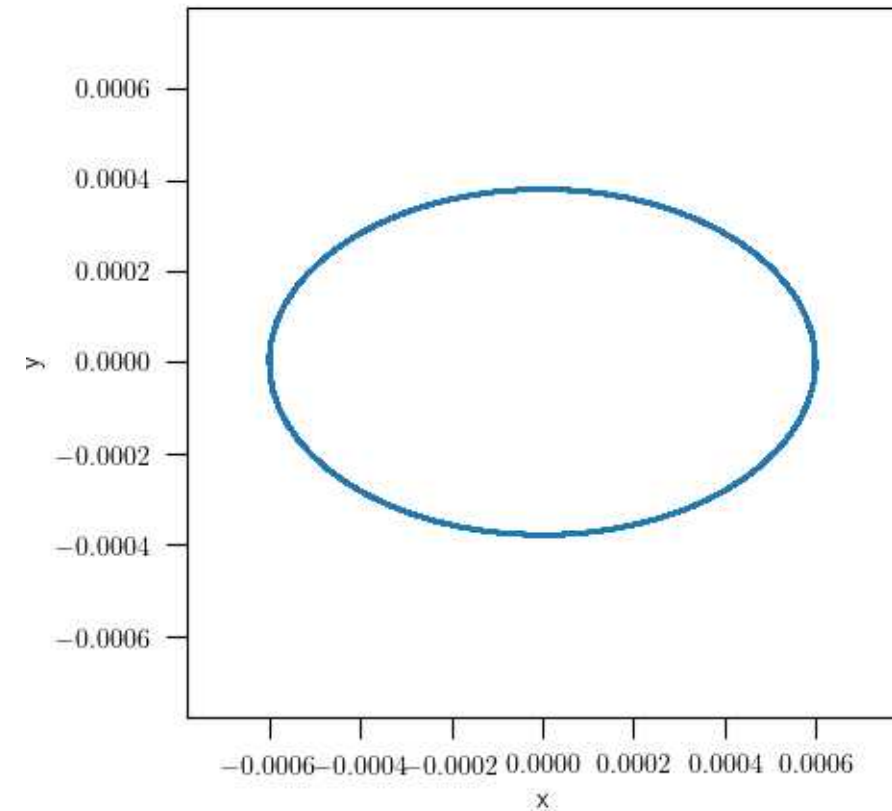
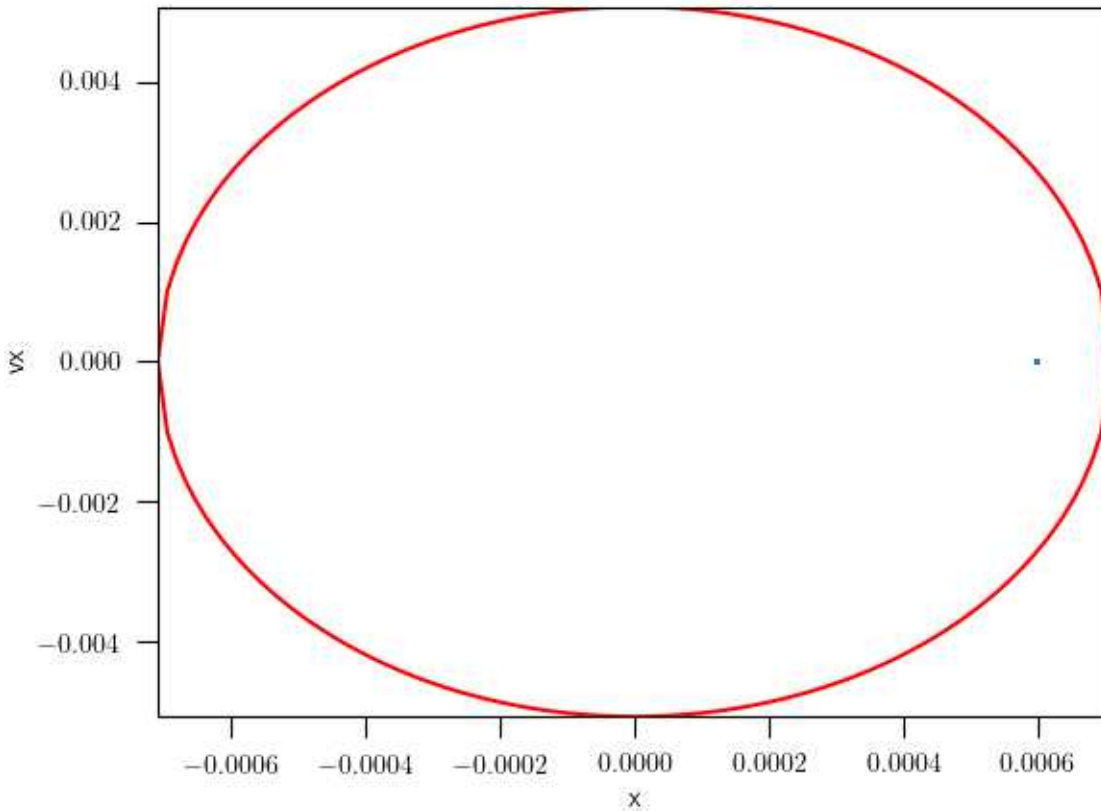


$$R \ll R_c$$

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --plotpotential  
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.965 --plotpotential
```

The flattening – frequency dependency

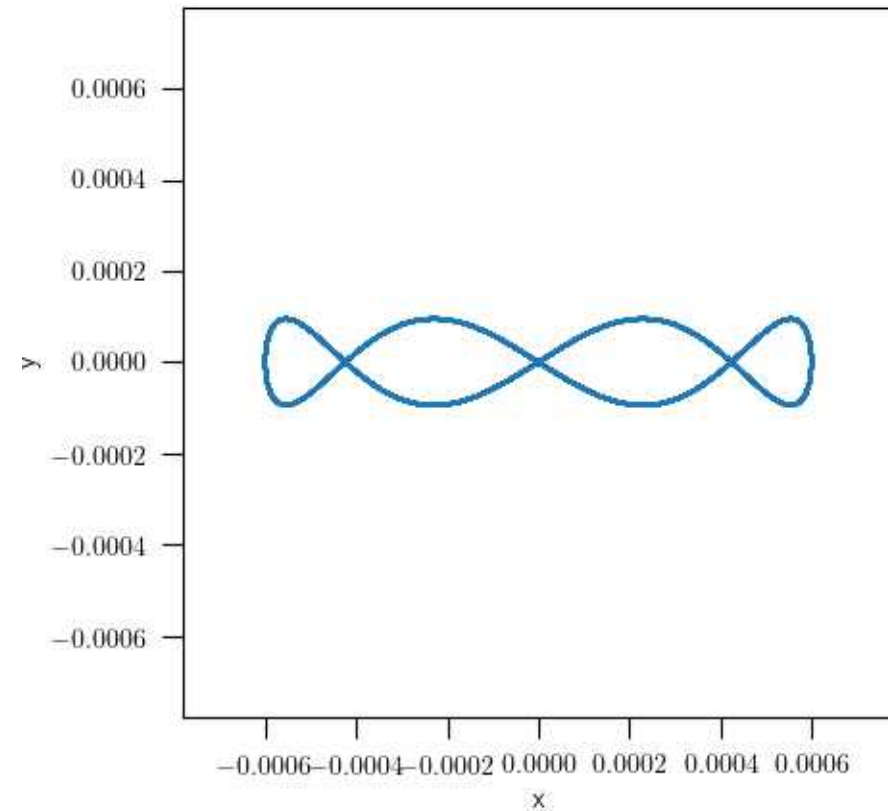
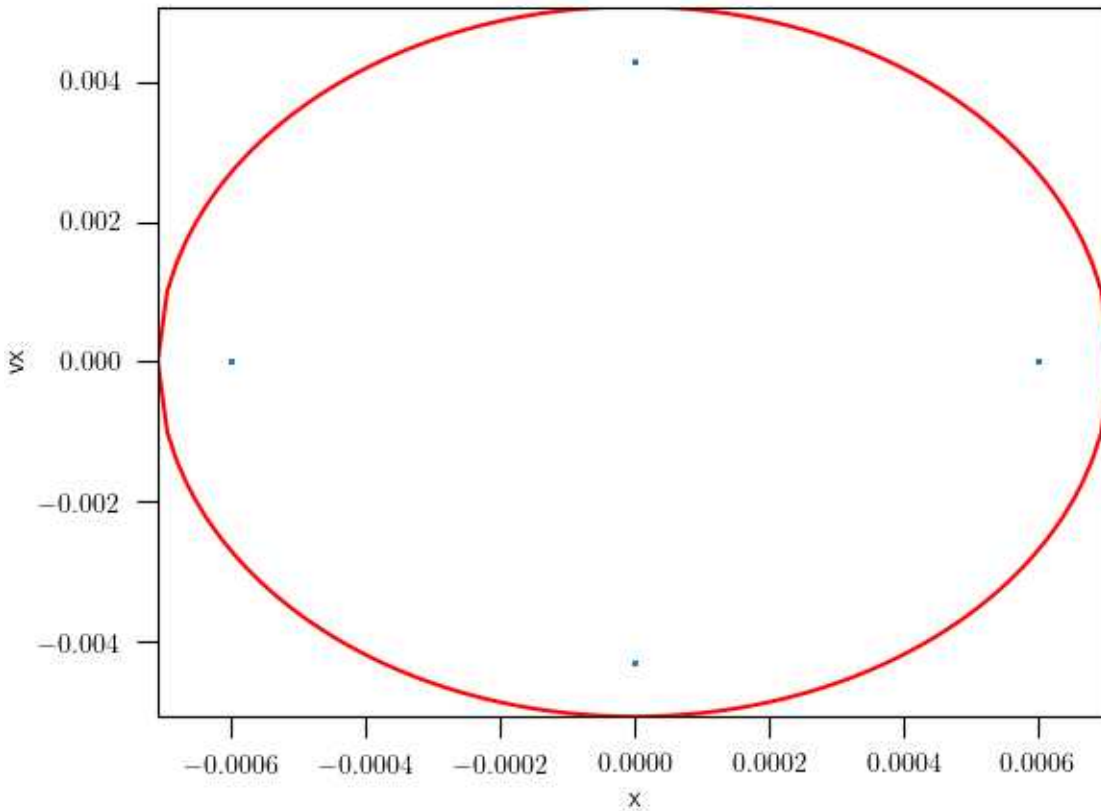
$$q = 1$$



```
./mapping.py --V0 1. --Rc 0.14 --q 1.0 -E -1.9661 --x 0.0006
```

The flattening – frequency dependency

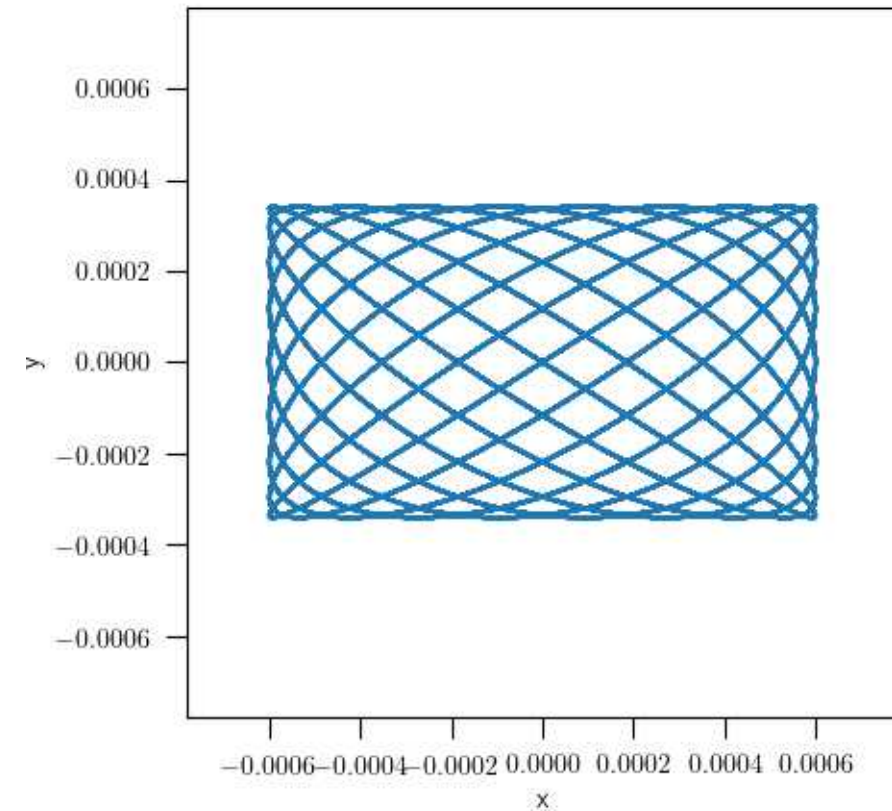
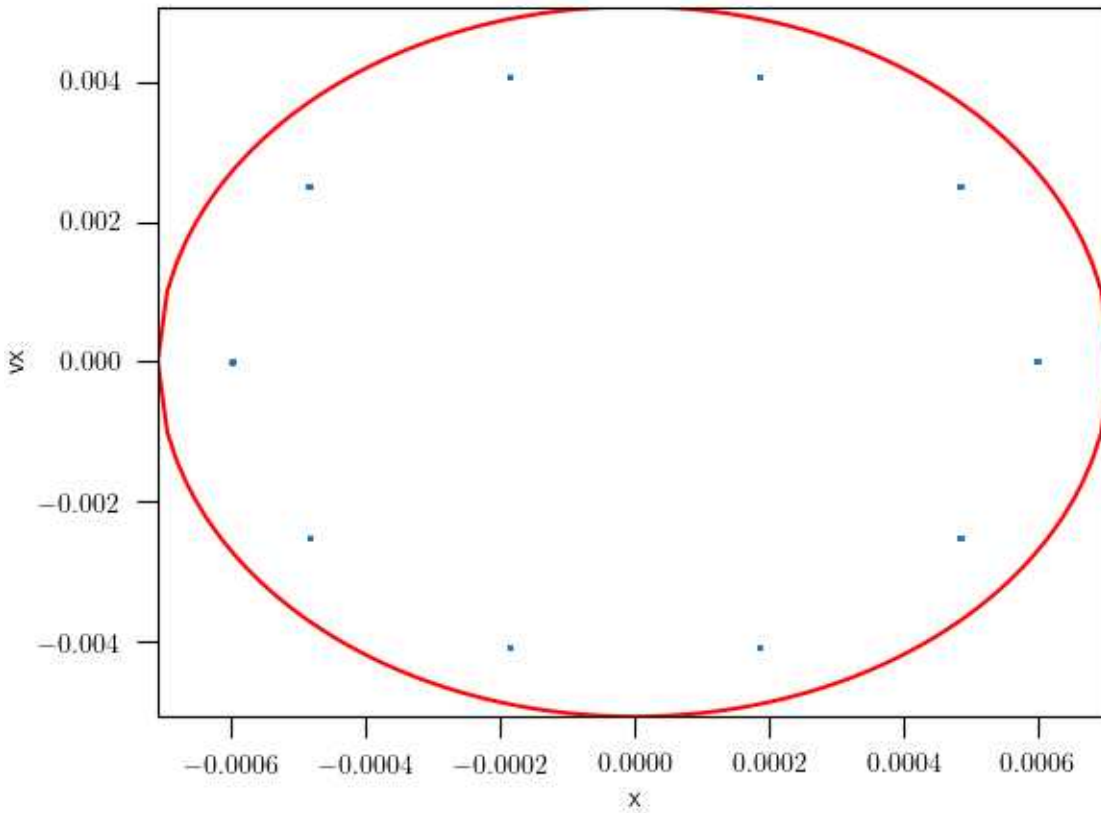
$$q = 0.25$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.25 -E -1.9661 --x 0.0006
```

The flattening – frequency dependency

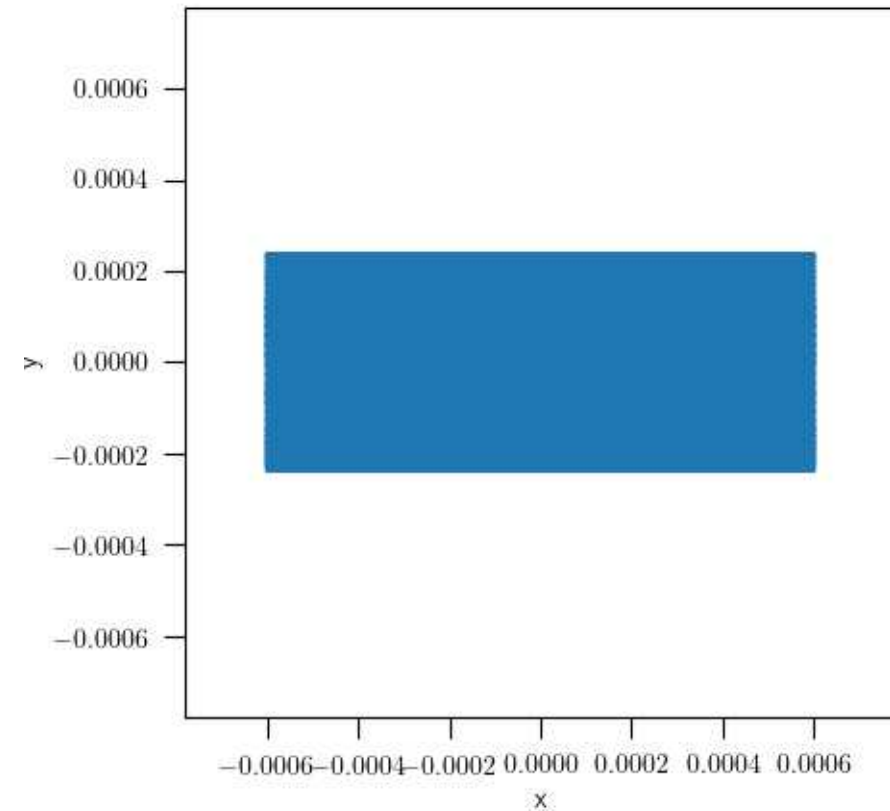
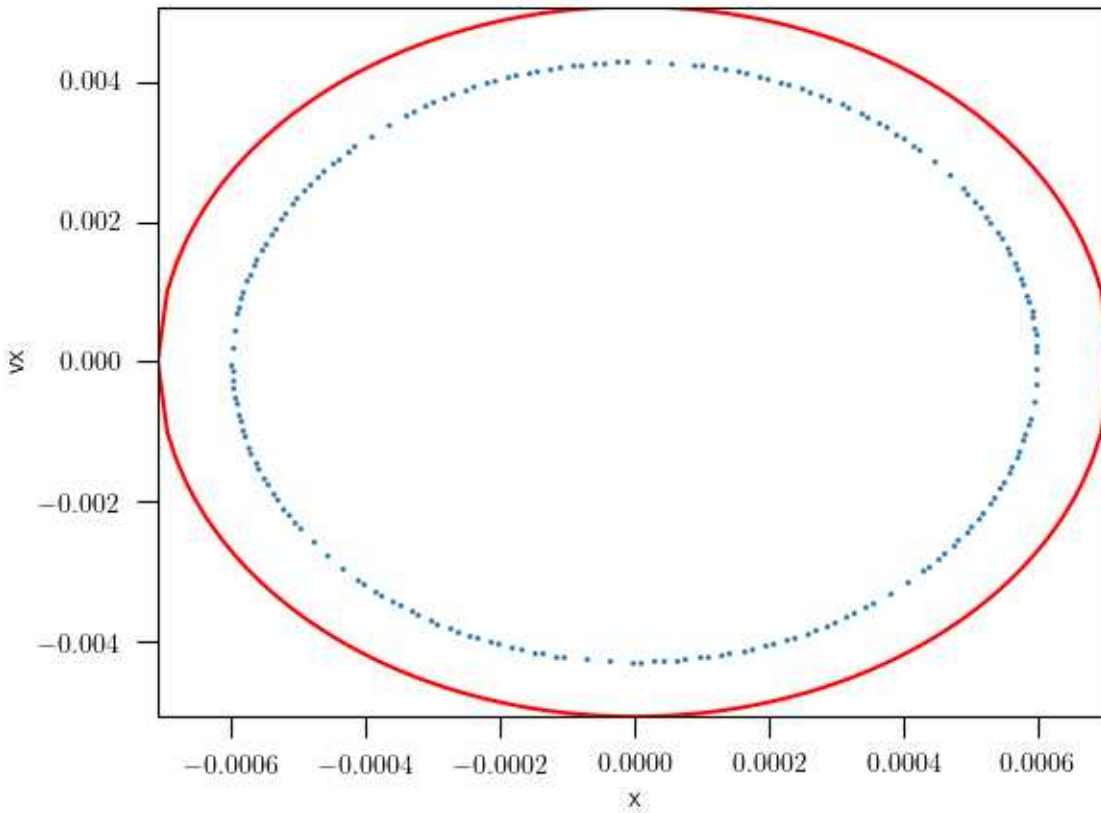
$$q = 0.9$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.9661 --x 0.0006
```

The flattening – frequency dependency

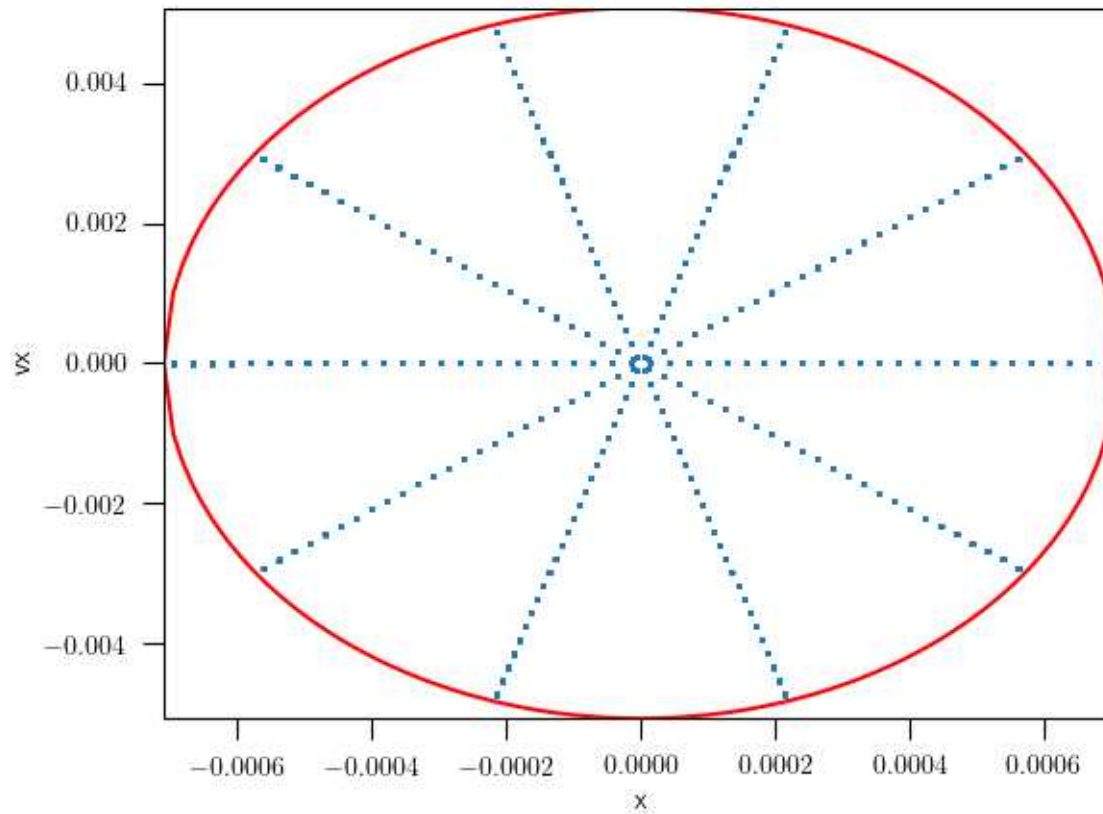
$$q = 0.62388462341$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.62388462341 -E -1.9661 --x 0.0006 --nlaps 200
```

“Complete” phase space

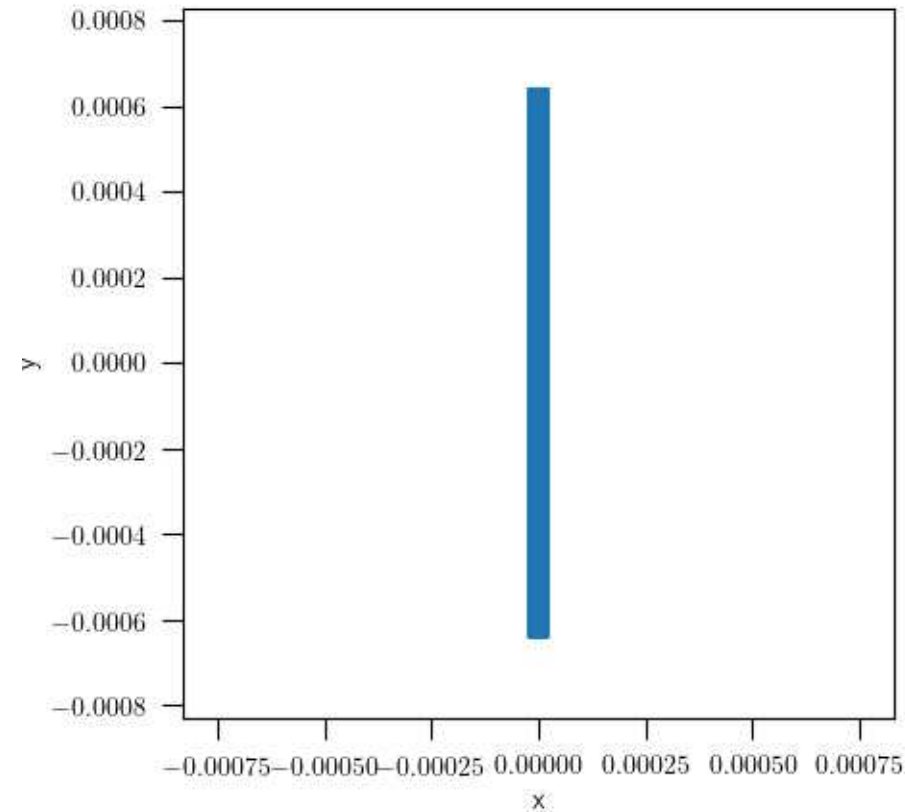
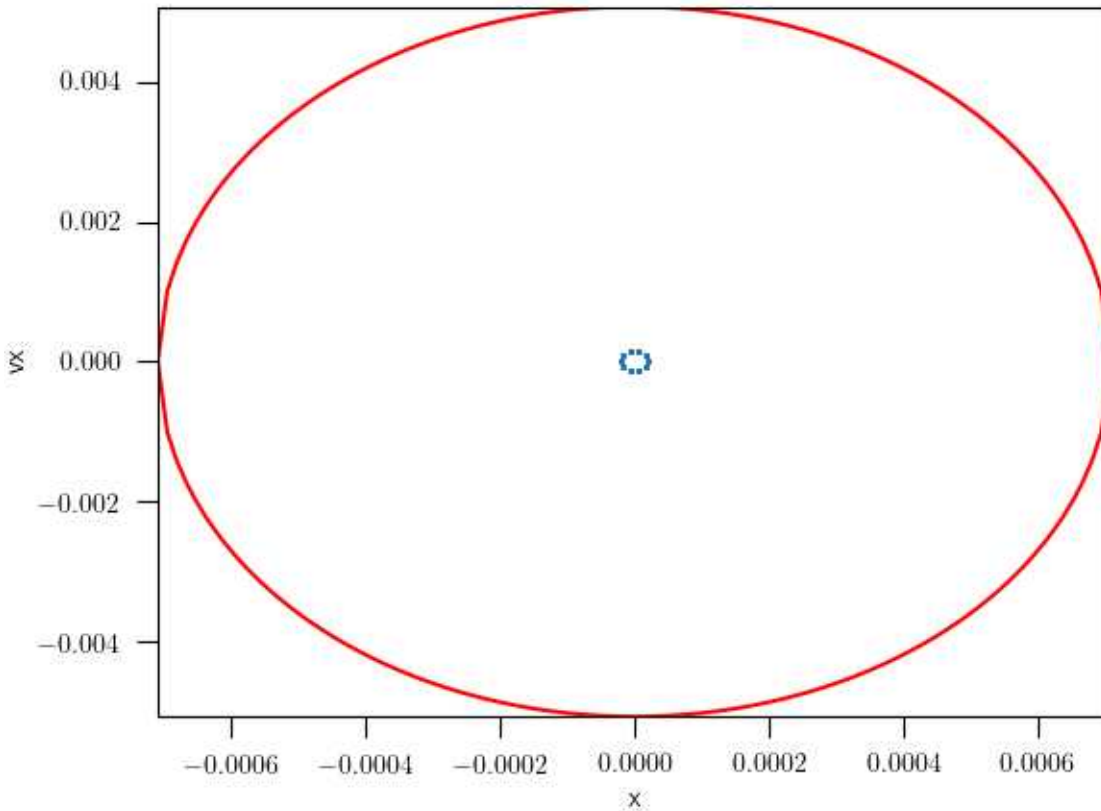
$$q = 0.9$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.9661 --norbits 50
```

small x, Y-elongated orbits (box orbit)

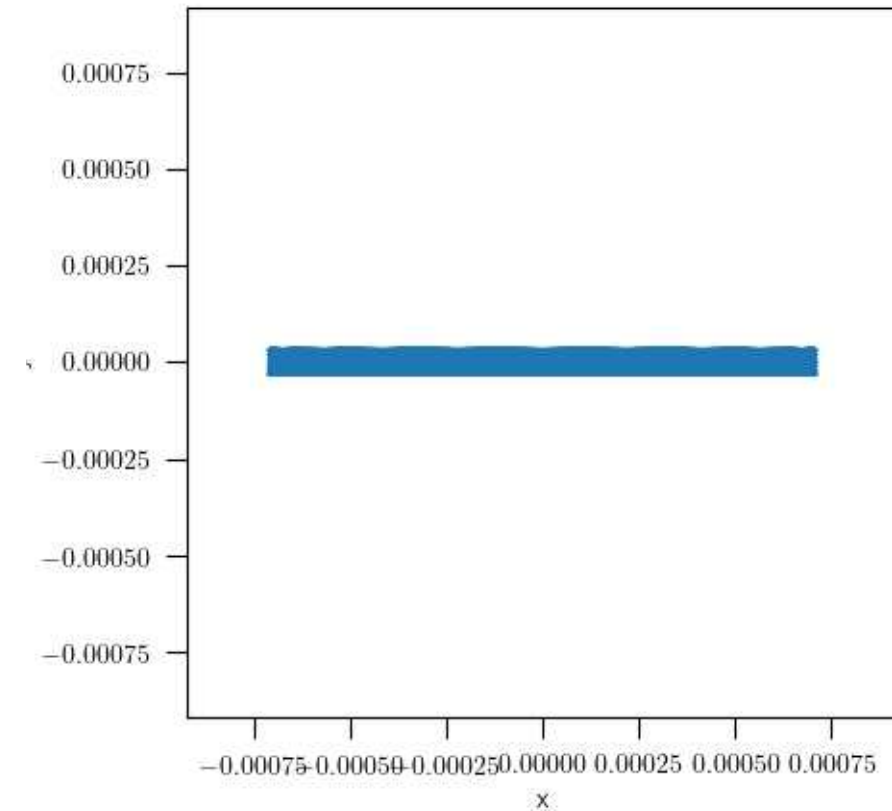
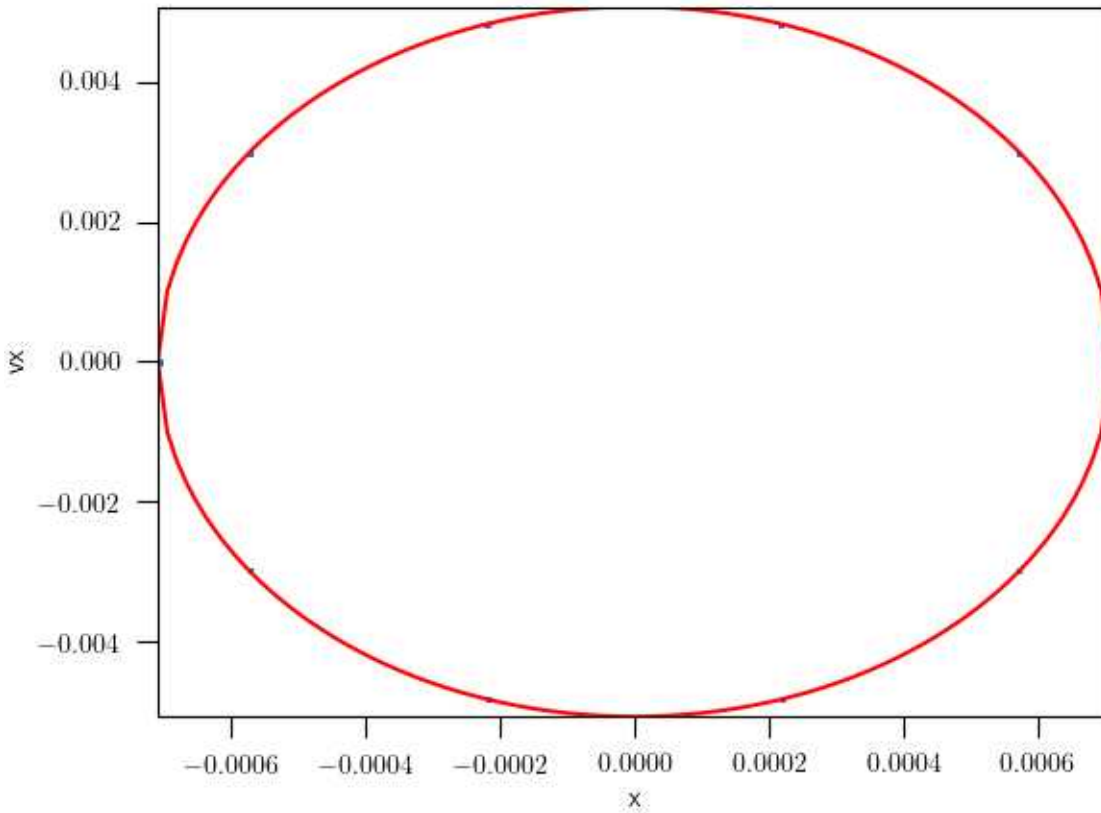
$$q = 0.9$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.9661 --x 0.00002
```

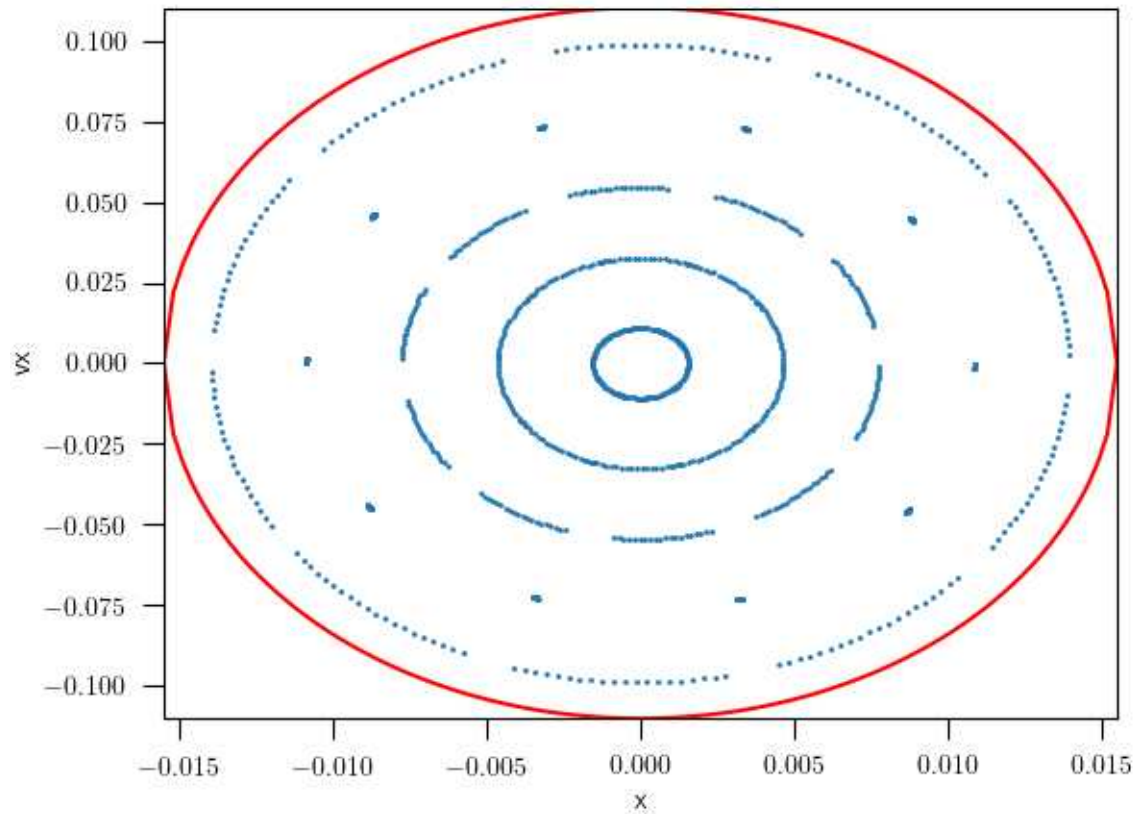
large x, X-elongated orbits (box orbit)

$$q = 0.9$$



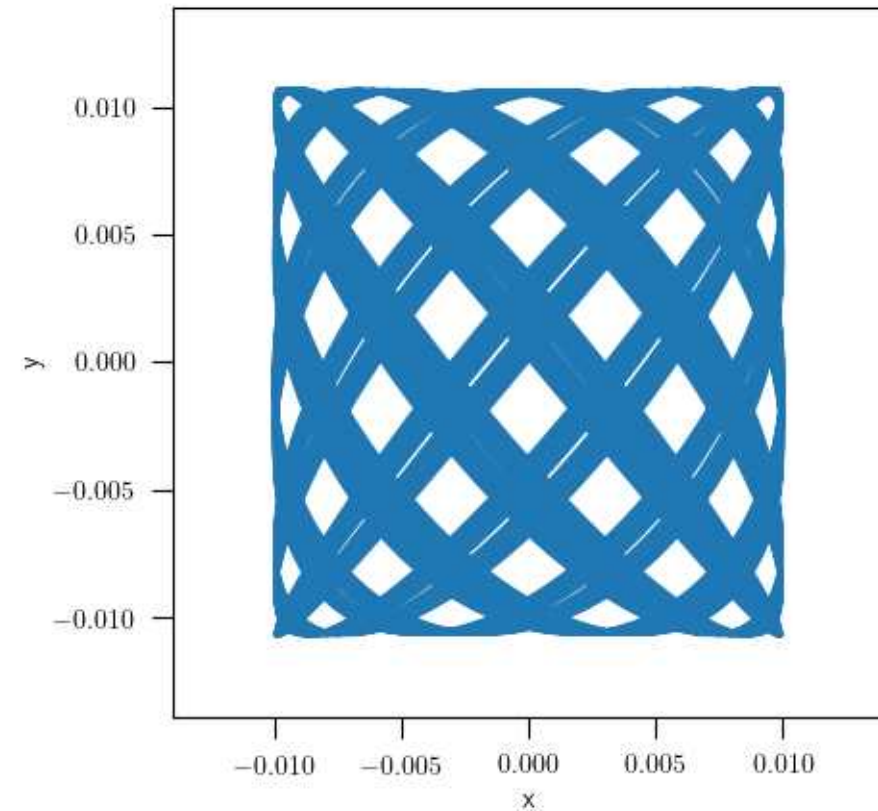
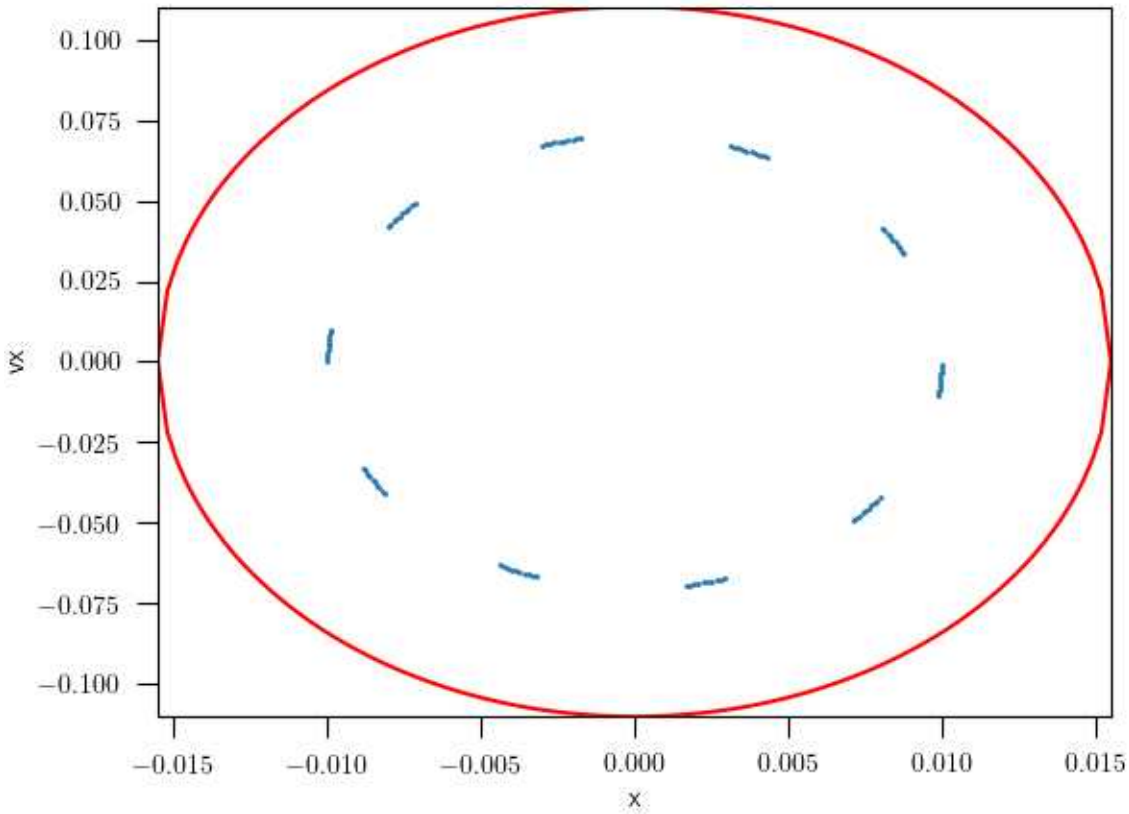
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.9661 --x 000709
```

Increasing energy : perturbed harmonic oscillator (coupling terms)



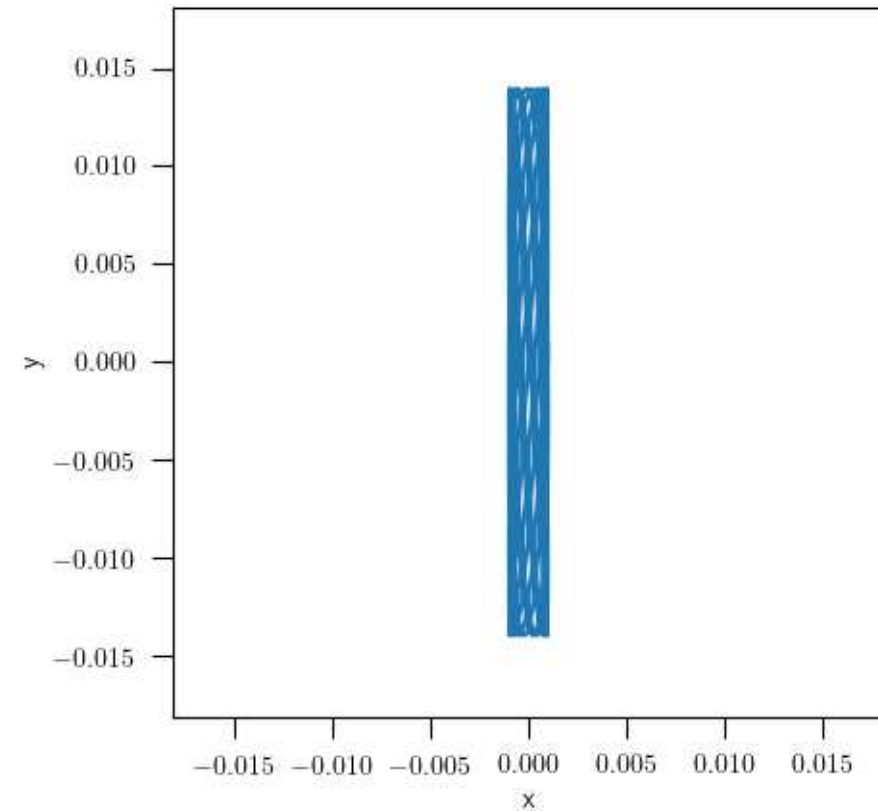
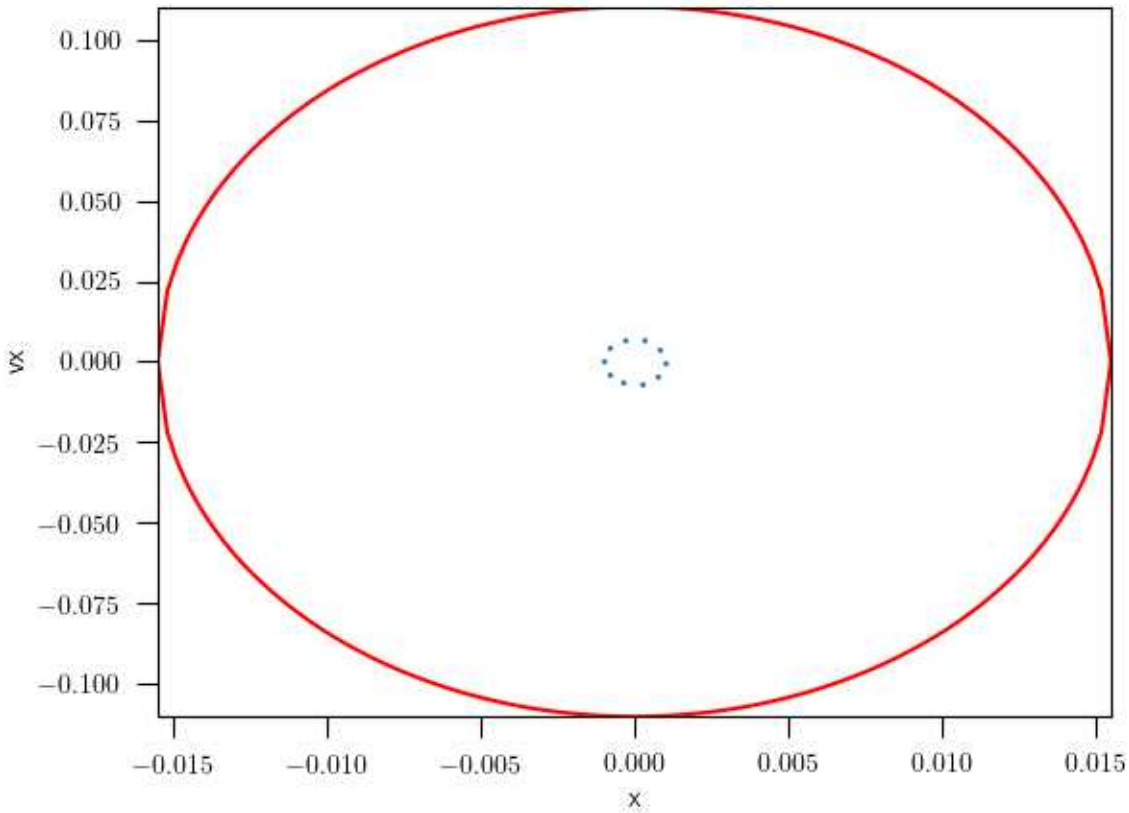
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.96
```

Increasing energy : perturbed harmonic oscillator



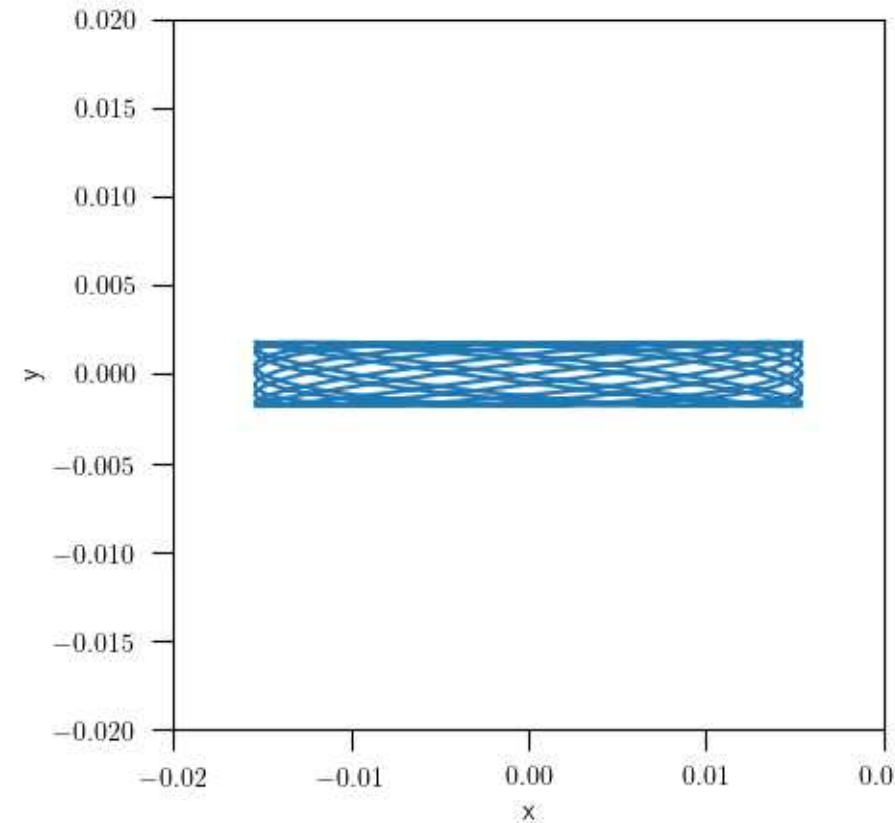
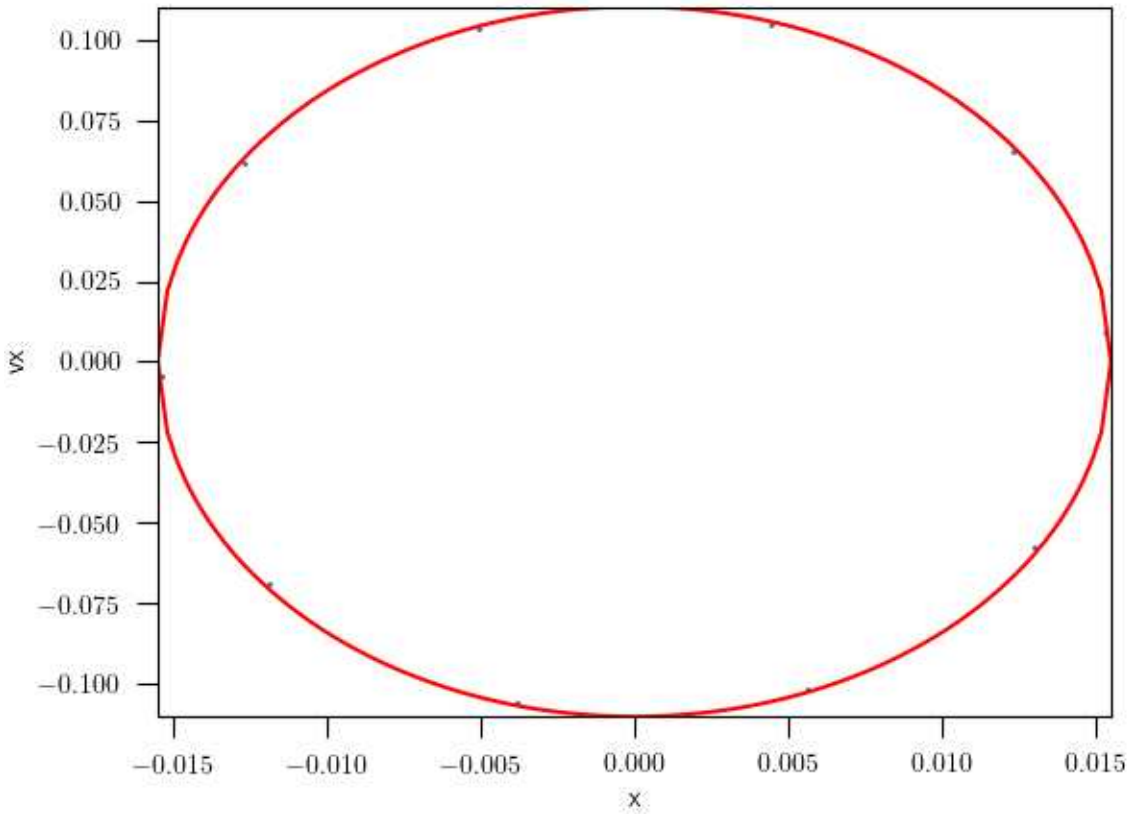
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.96 --x 0.01
```

small x, Y-elongated orbits (box orbit)



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.96 --x 0.001 --nlaps 10
```

large x, X-elongated orbits (box orbit)



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.96 --x 0.0154 --nlaps 10
```

$$R \gg R_c$$

Motions for $R \gg R_c$

$$\begin{aligned}\phi(x, y) &= \frac{1}{2} V_0^2 \ln \left(R_c + x^2 + \frac{y^2}{q^2} \right) \\ &\approx \frac{1}{2} V_0^2 \ln \left(x^2 + \frac{y^2}{q^2} \right) \sim \frac{1}{2} V_0^2 \ln (R^2) \\ &\quad q \approx 1\end{aligned}$$

Orbit families

① box orbits

(disturbed 2D harmonic oscillator)

$$V_{//} : V_y \approx 0$$



if $V_y = 0$: radial orbit ($L_z = 0$)

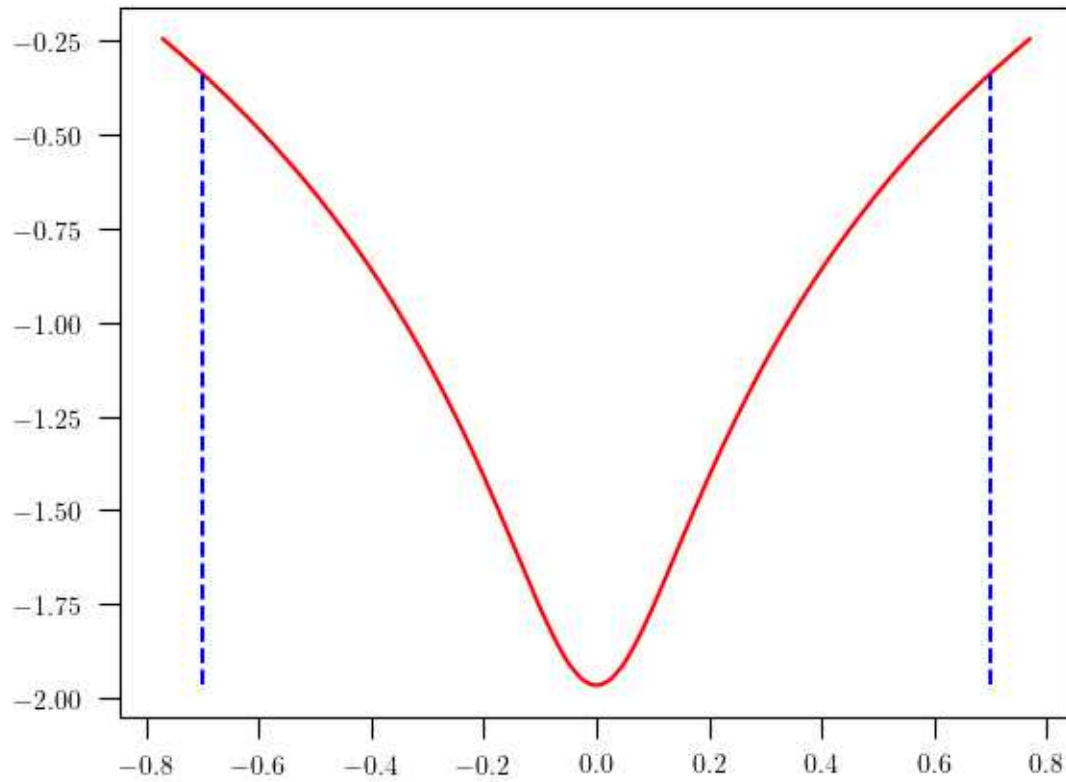
② loop orbits

$$V_{\perp} : V_y \approx V_0$$



if $V_y = V_0$: circular orbit
 $q = 1$

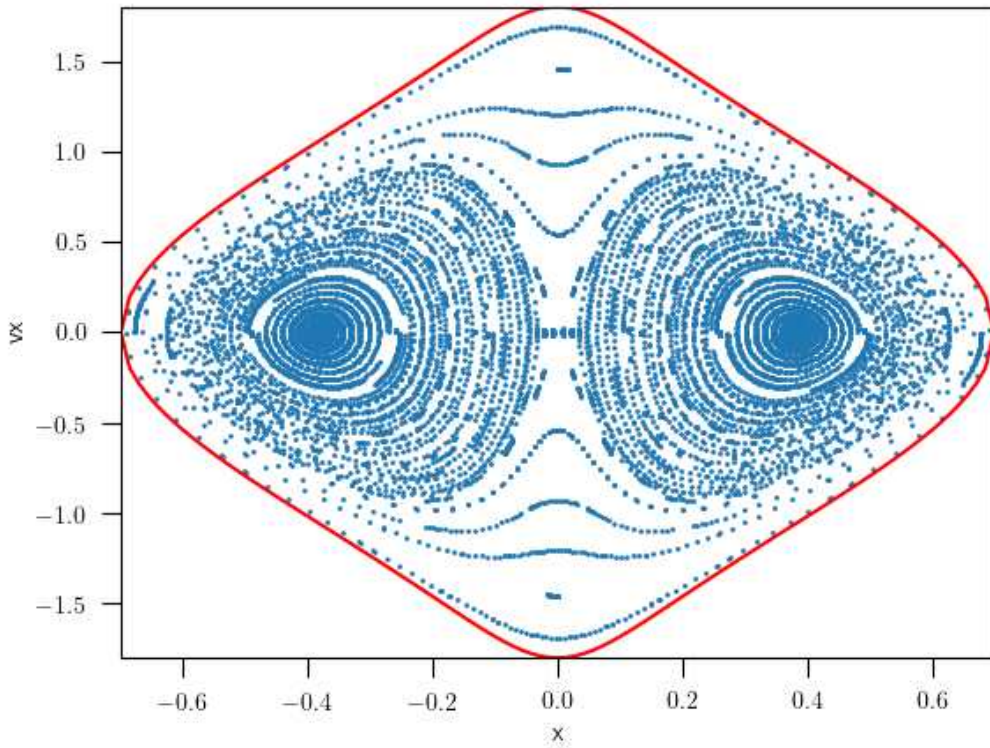
Potential and energy



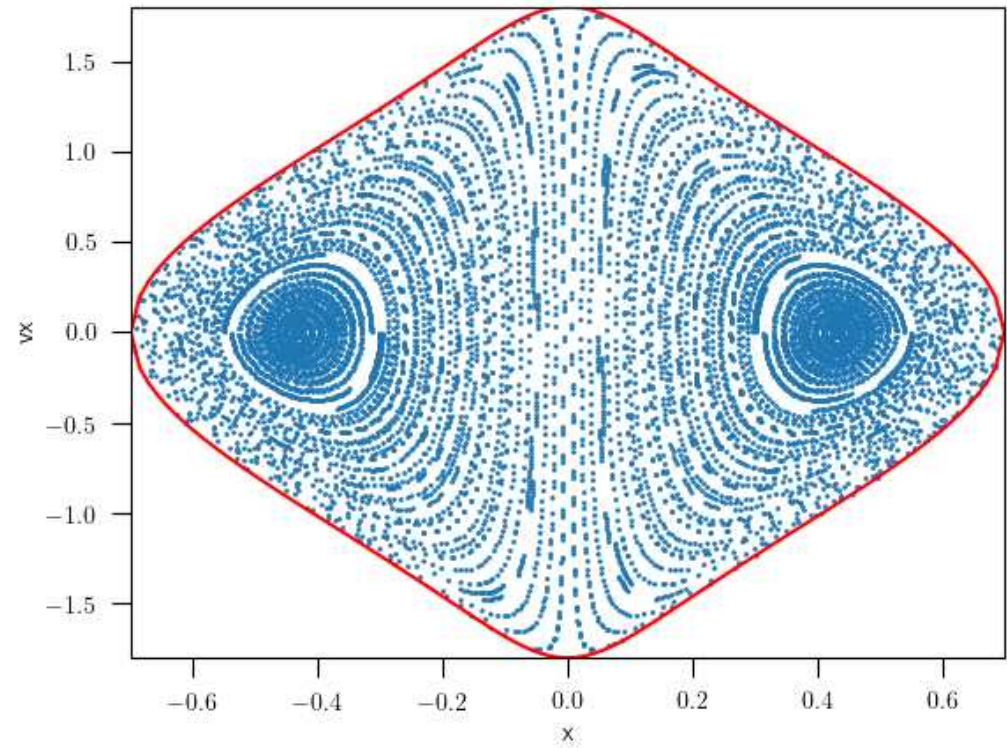
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --plotpotential
```

Phase space

$$q = 0.9$$



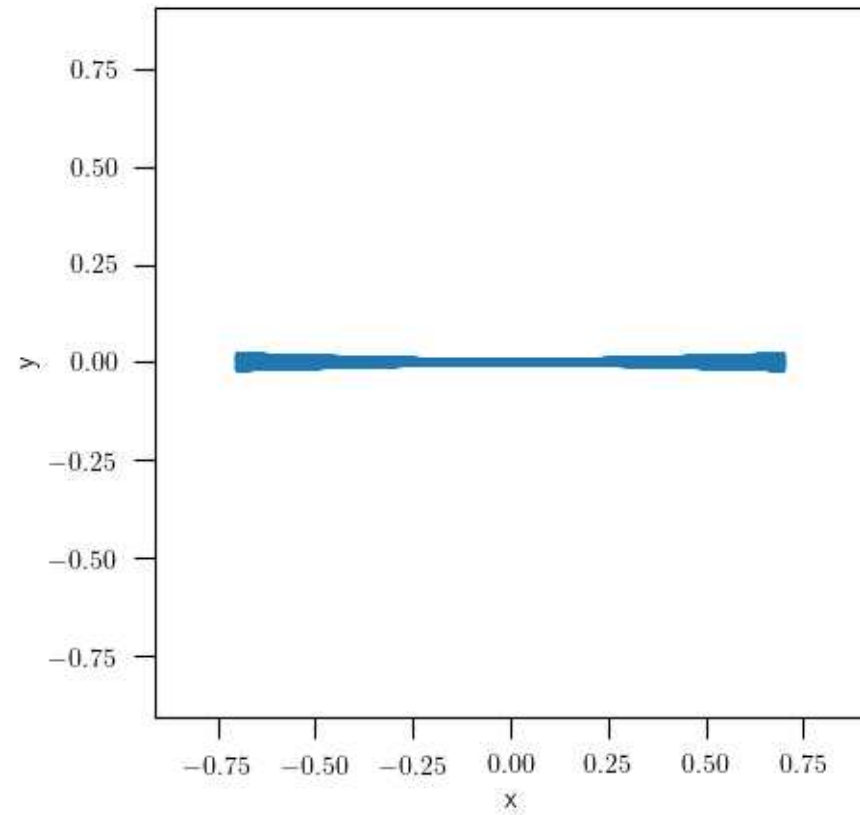
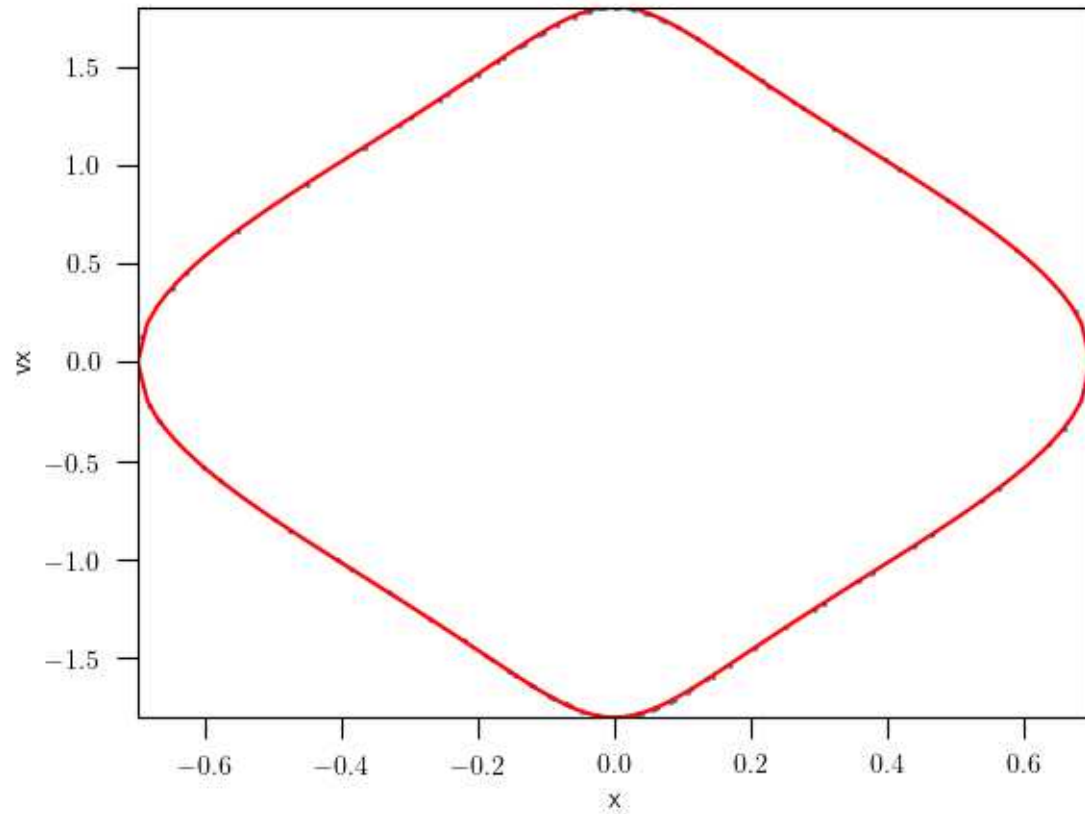
$$q = 1.0$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --norbits 100
```

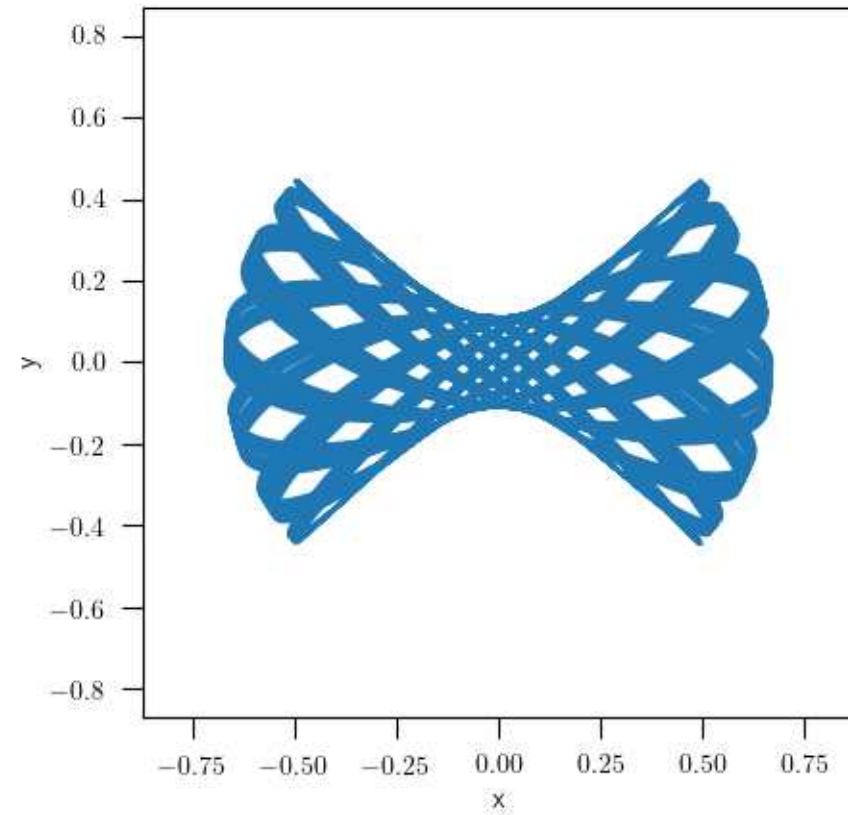
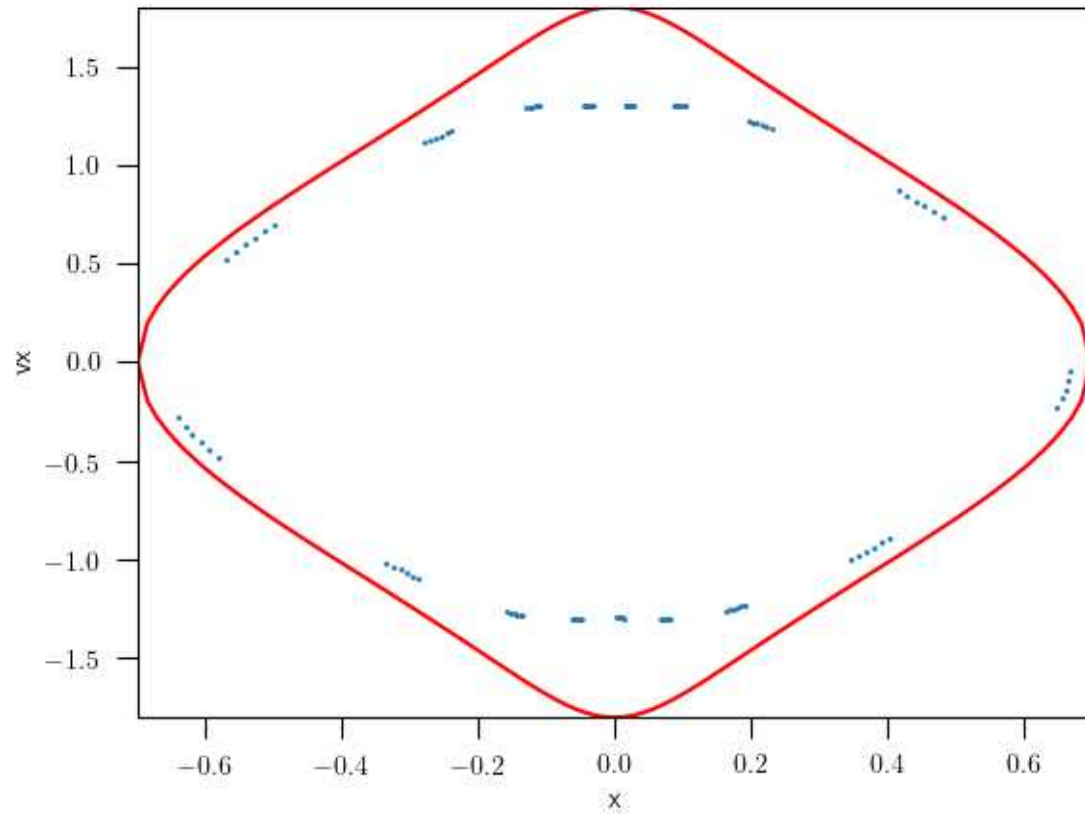
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```

Box orbits



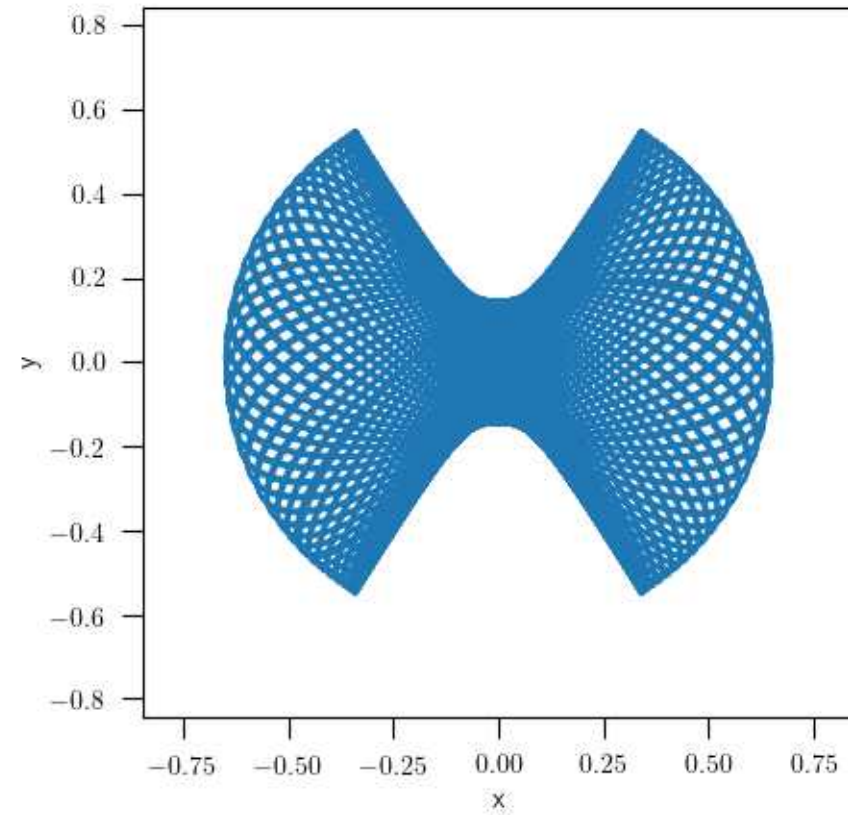
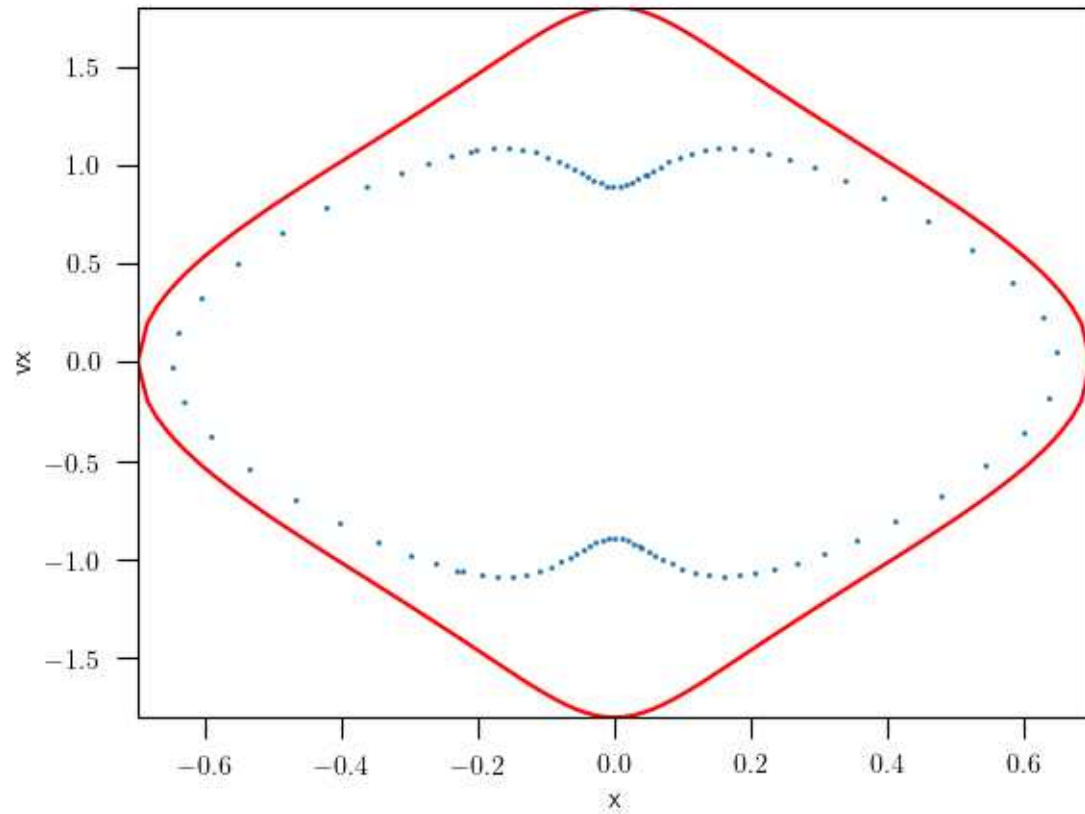
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.7
```

Box orbits



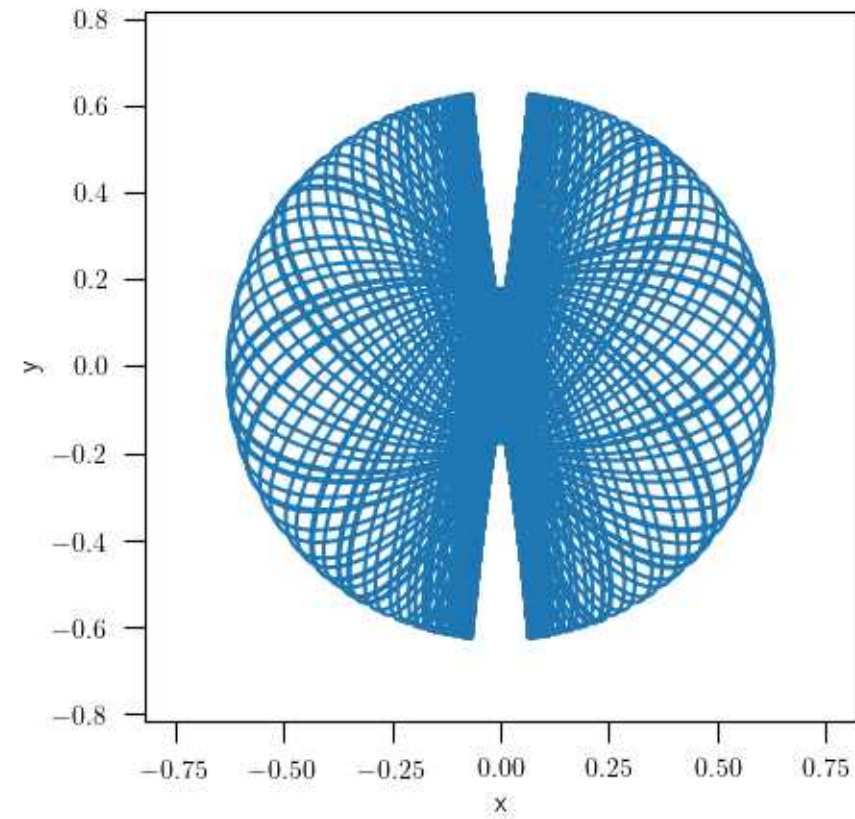
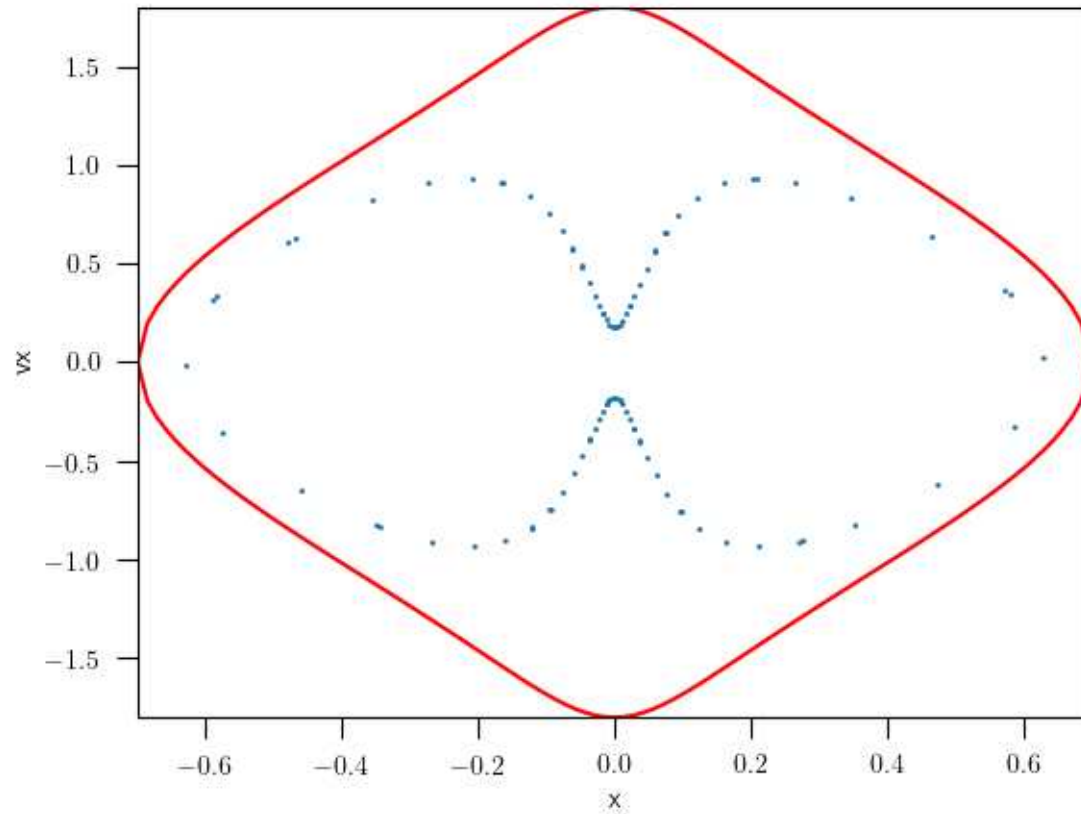
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --x 0.67
```

Box orbits



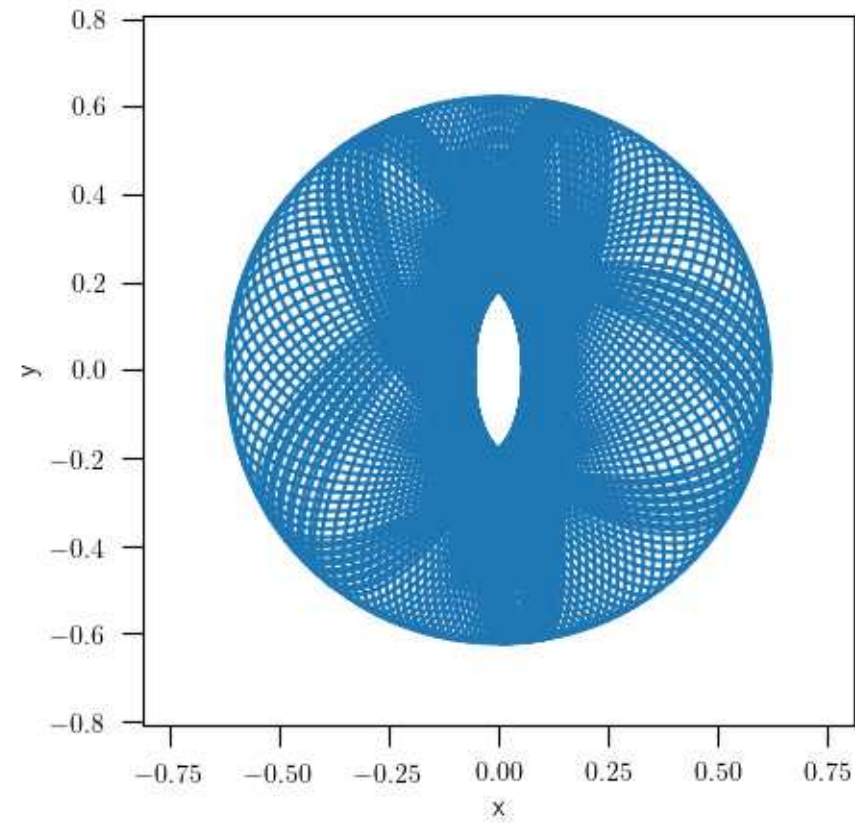
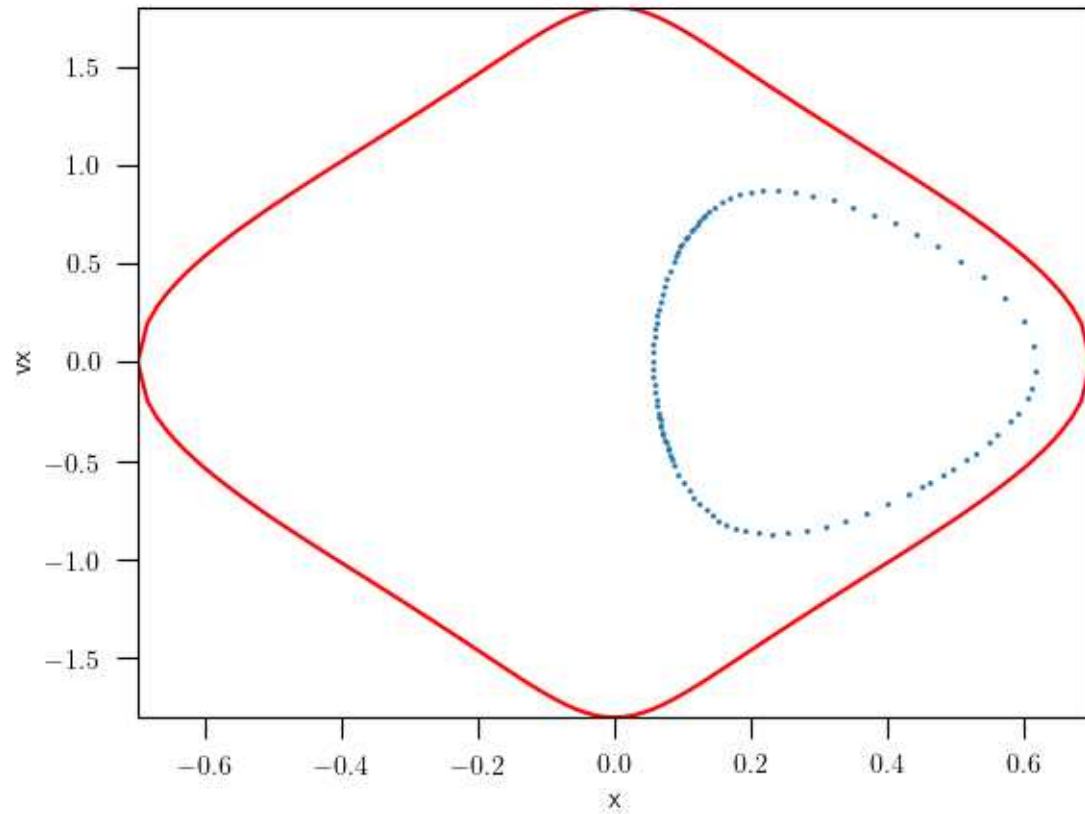
```
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```

Box orbits



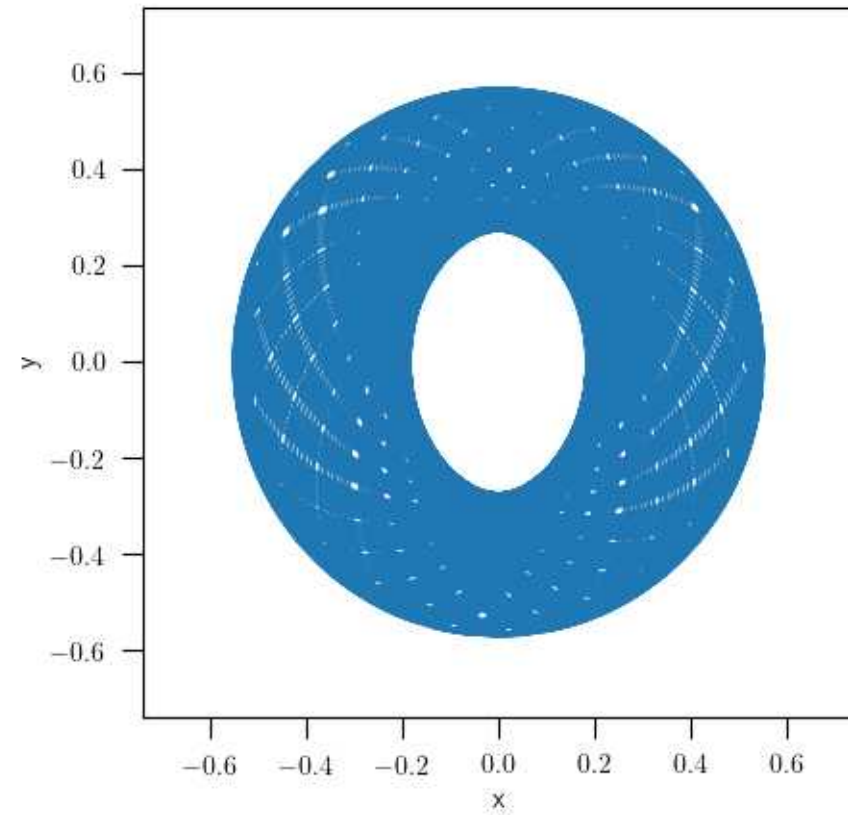
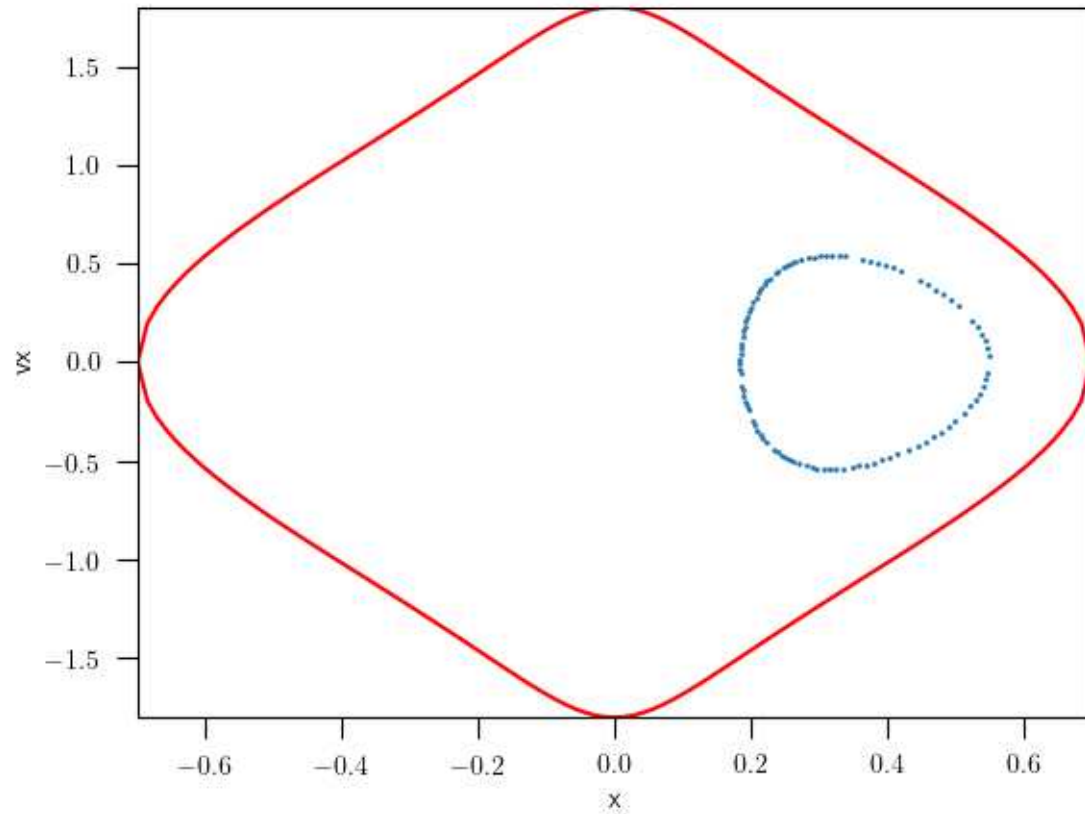
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.63
```

Loop orbits



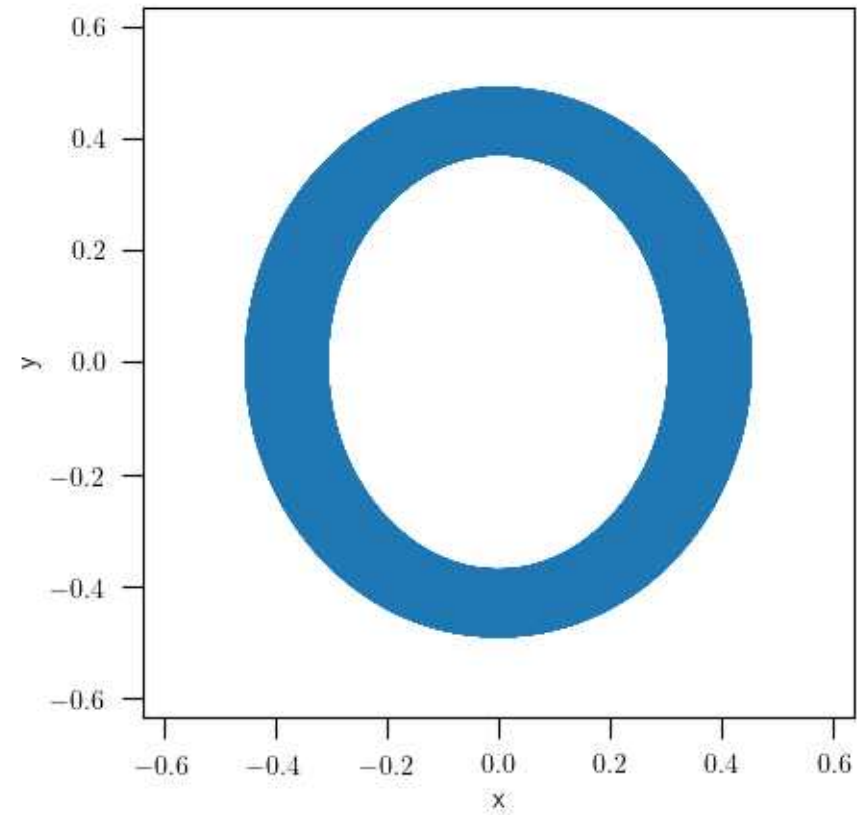
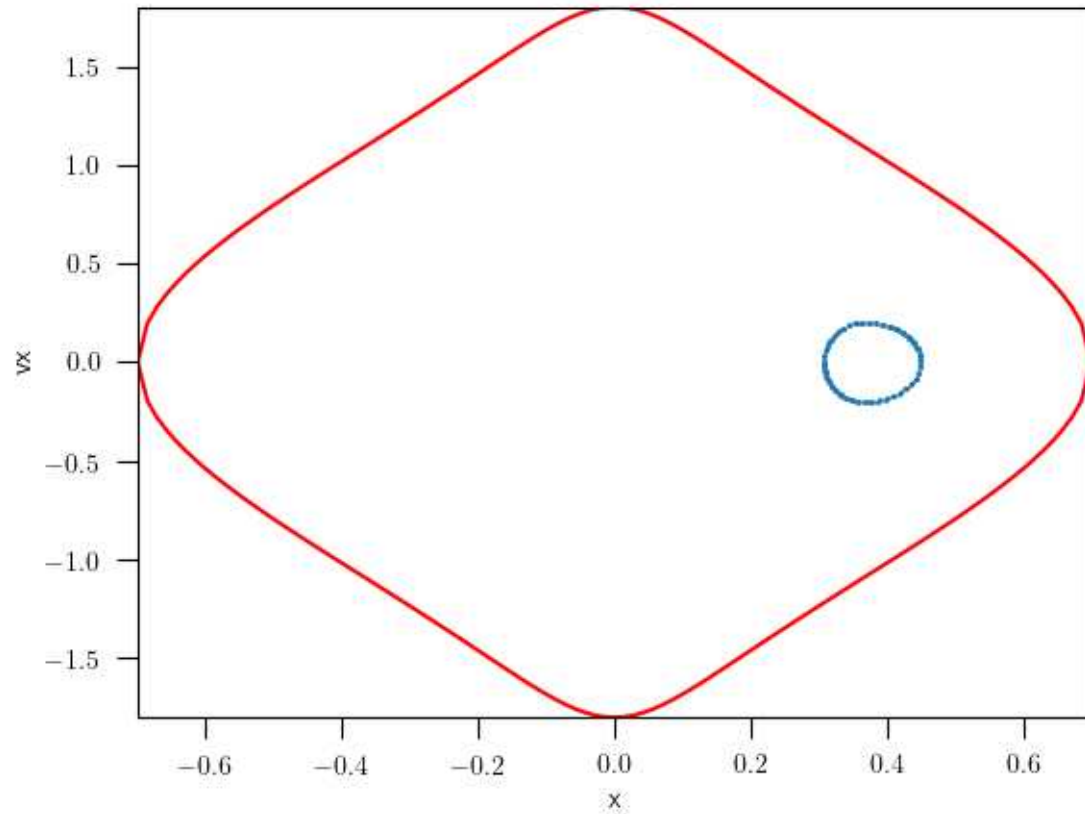
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.62
```

Loop orbits



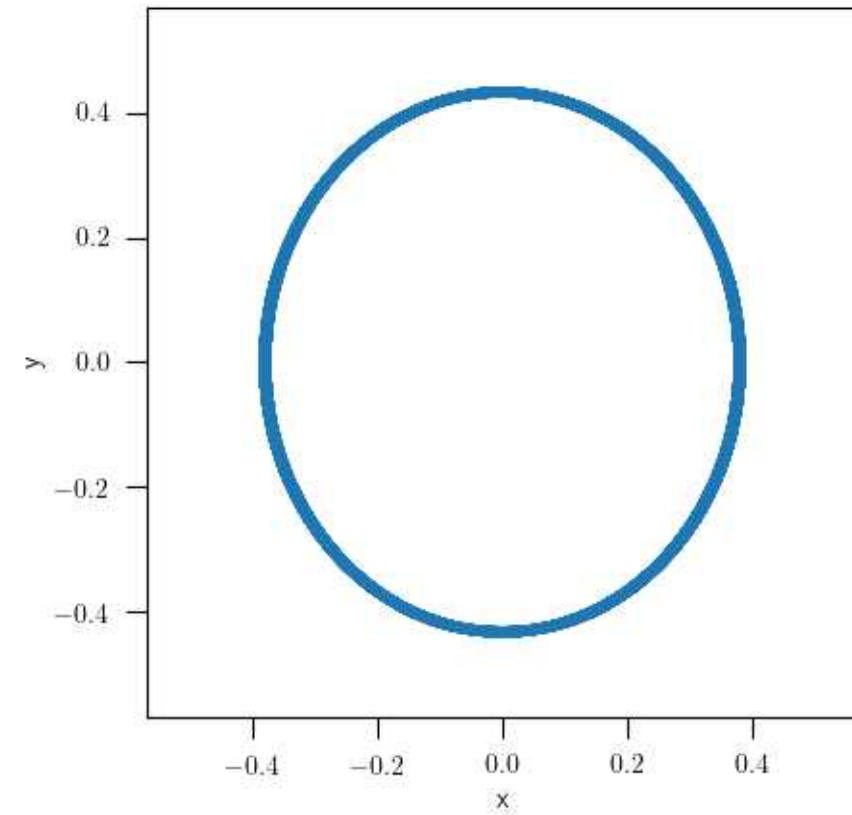
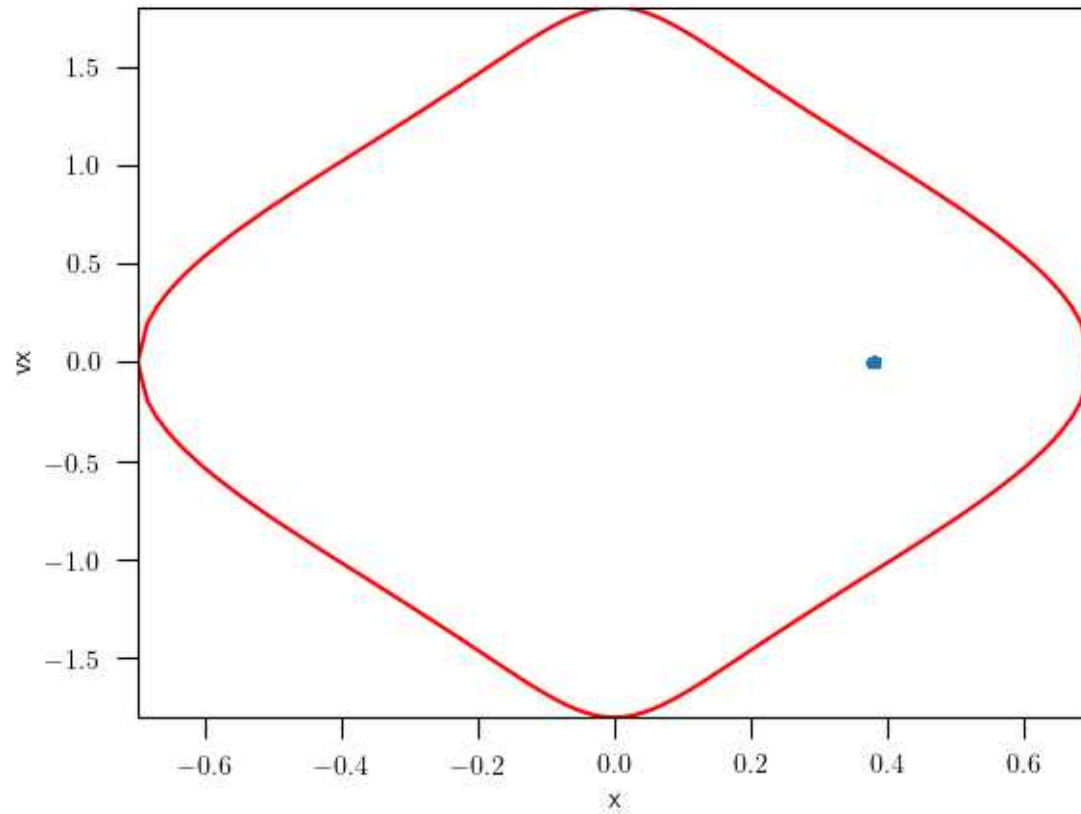
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.55
```

Loop orbits



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --x 0.45
```

Loop orbits

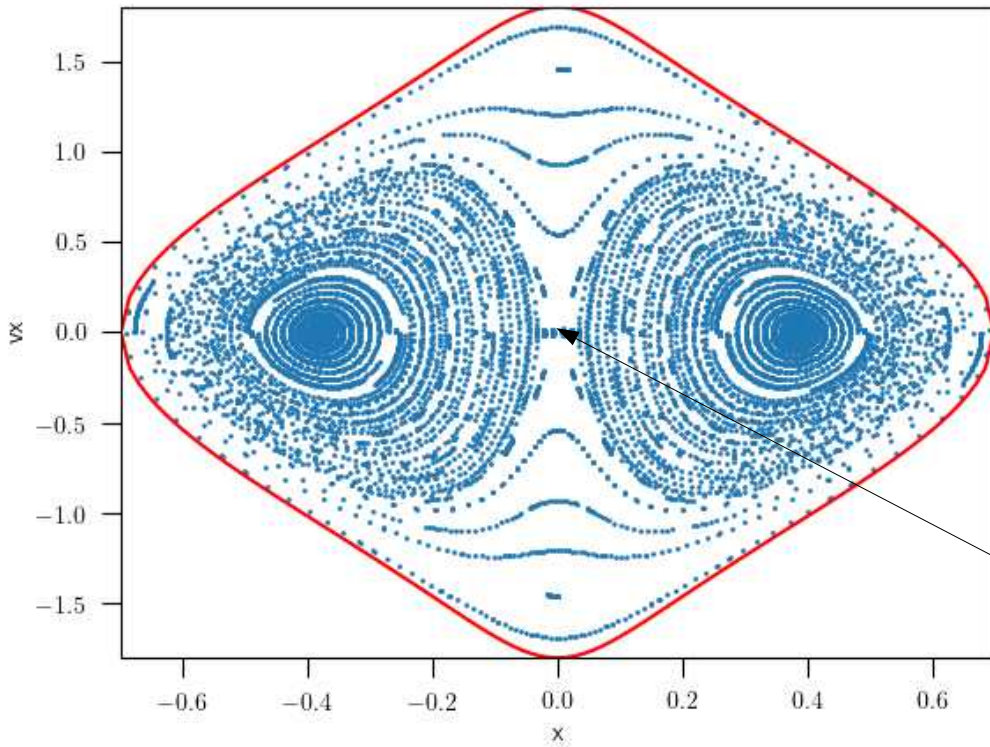


```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.374
```

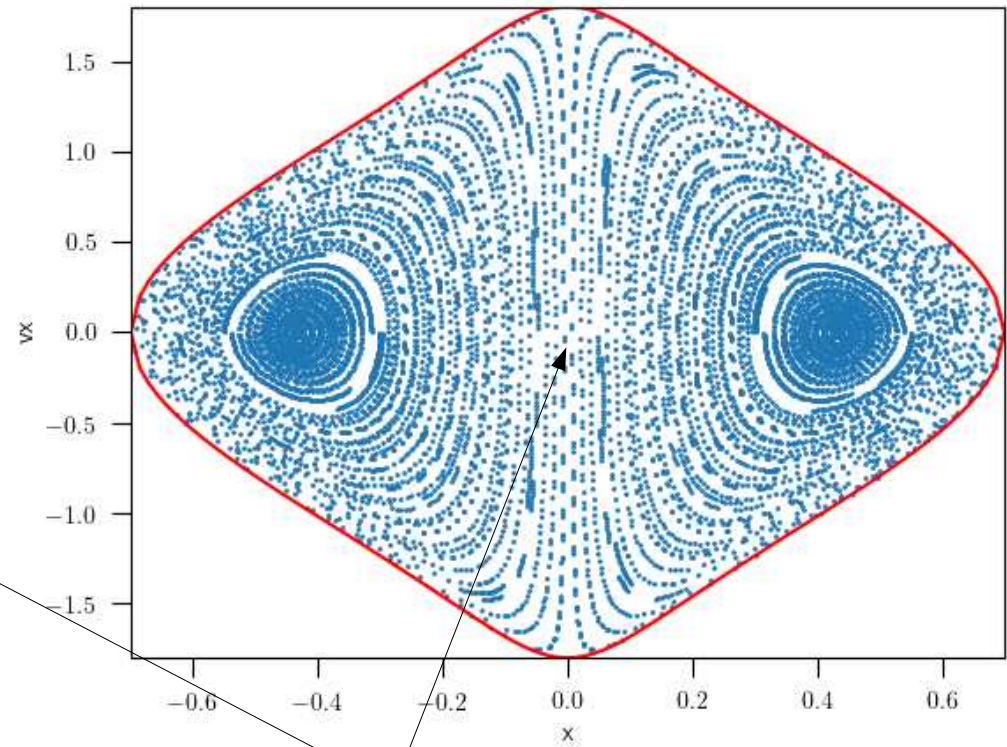
Box orbits elongated towards the y axis

$$q = 0.9$$

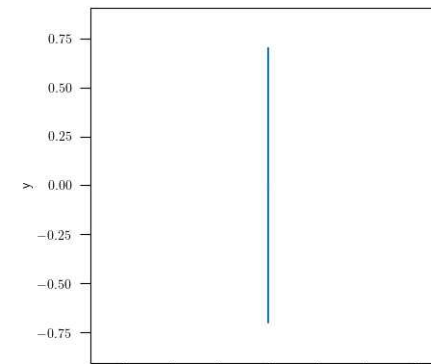
$$q = 1.0$$



unstable



stable

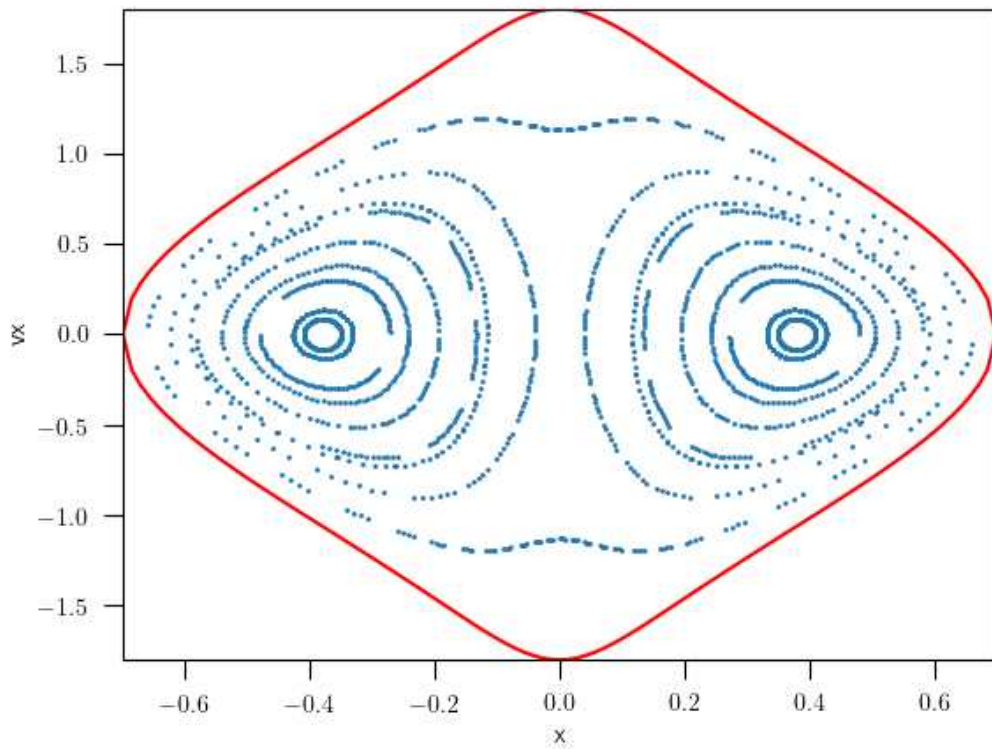


```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --norbits 100  
./mapping.py --V0 1. --Rc 0.14 --q 1.0 -E -0.337 --norbits 100
```

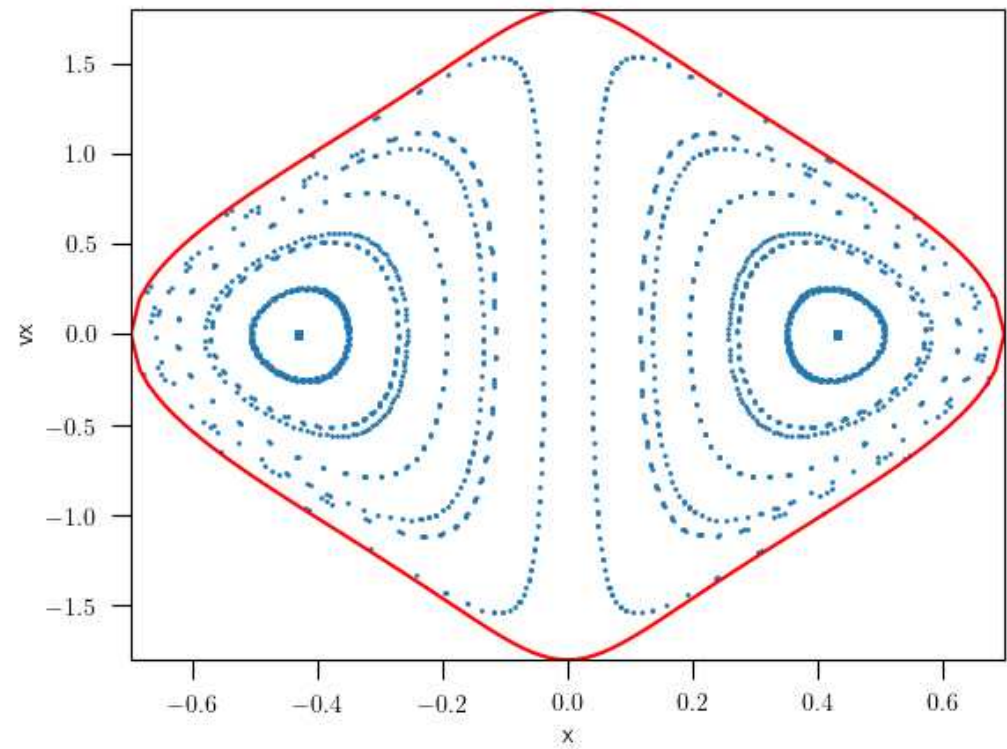
Integral of motions ?

Integral of motions ?

$$q = 0.9$$



$$q = 1.0$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --norbits 18
```

```
./mapping.py --V0 1. --Rc 0.14 --q 1.0 -E -0.337 --norbits 18
```

Integrals of motions

① "nearly circular orbits"

Angular momentum conservation

$$L_z = x\dot{y} - y\dot{x}$$

can we compute $x = x(\dot{x})$ in the plane $y = 0$?

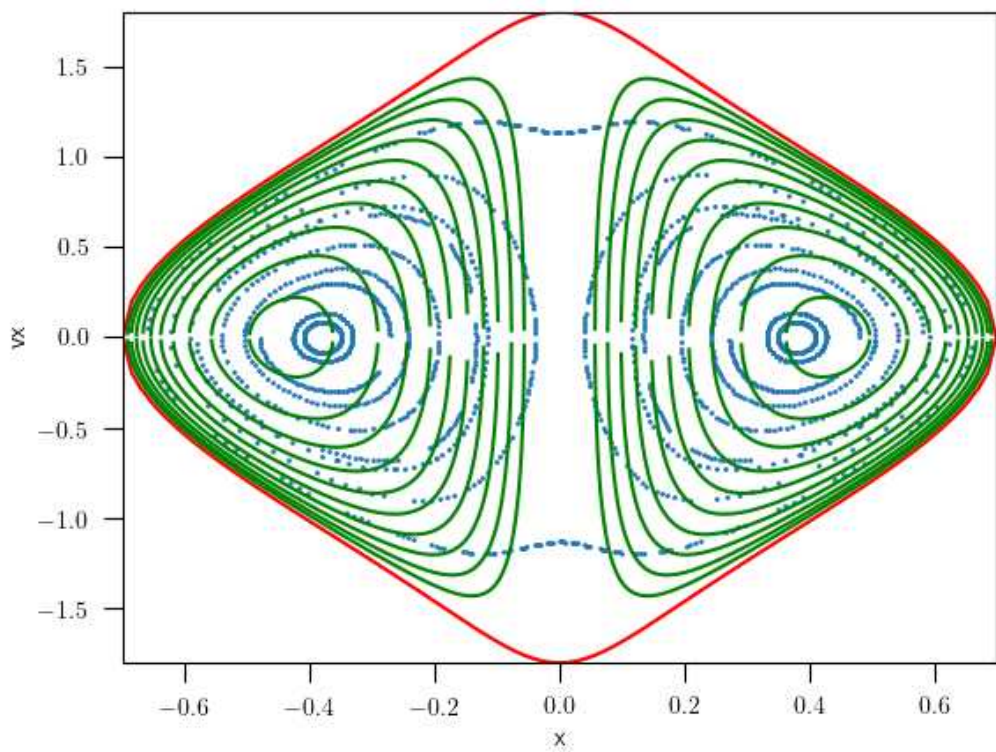
$$L_z = x\dot{y}$$

$$E = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 + \phi(x, y=0)$$

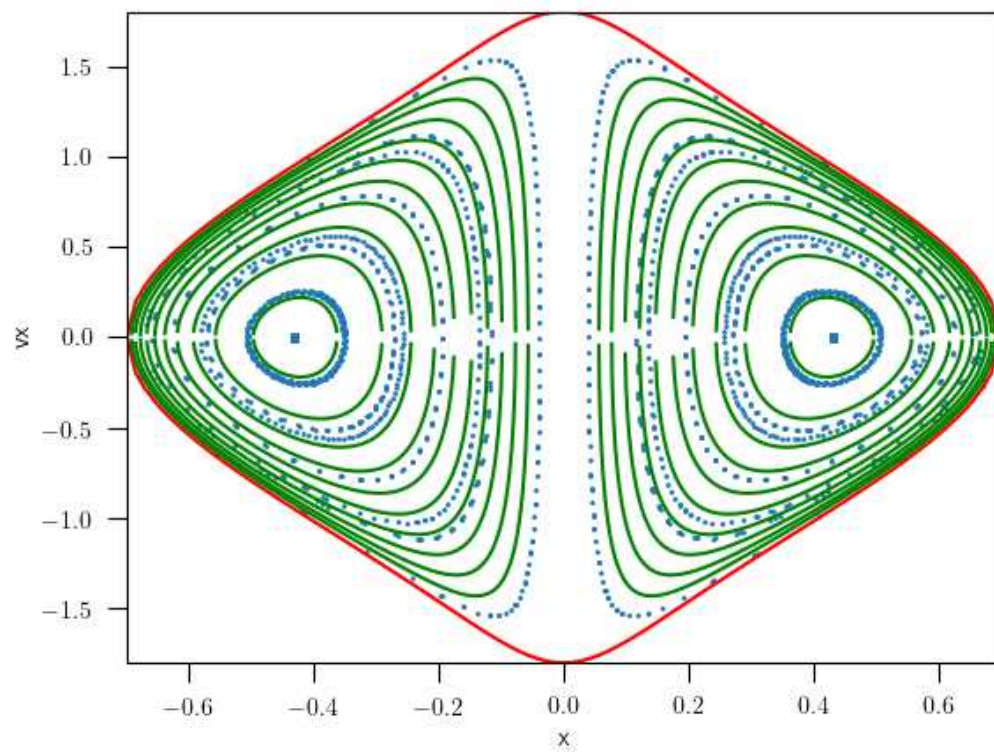
$$\dot{x} = \sqrt{2(E - \phi) - \dot{y}^2} = \sqrt{2\left(E - \phi - \frac{L_z^2}{x^2}\right)}$$

$$L_z$$

$$q = 0.9$$



$$q = 1.0$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --norbits 18 --add_ILz
```

```
./mapping.py --V0 1. --Rc 0.14 --q 1.0 -E -0.337 --norbits 18 --add_ILz
```

Integrals of motions

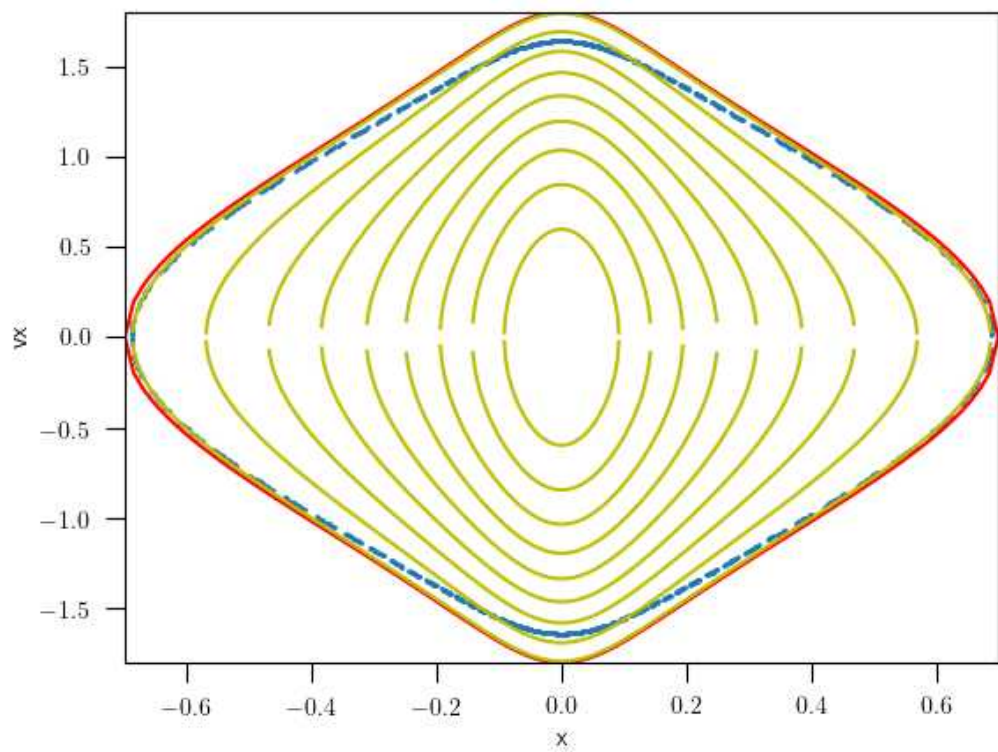
② Motion parallel to the long axis ($y = \dot{y} = 0$)

$$H_x = \frac{1}{2} \dot{x}^2 + \phi(x, y=0) = E_x \quad (\text{harmonic oscillator})$$

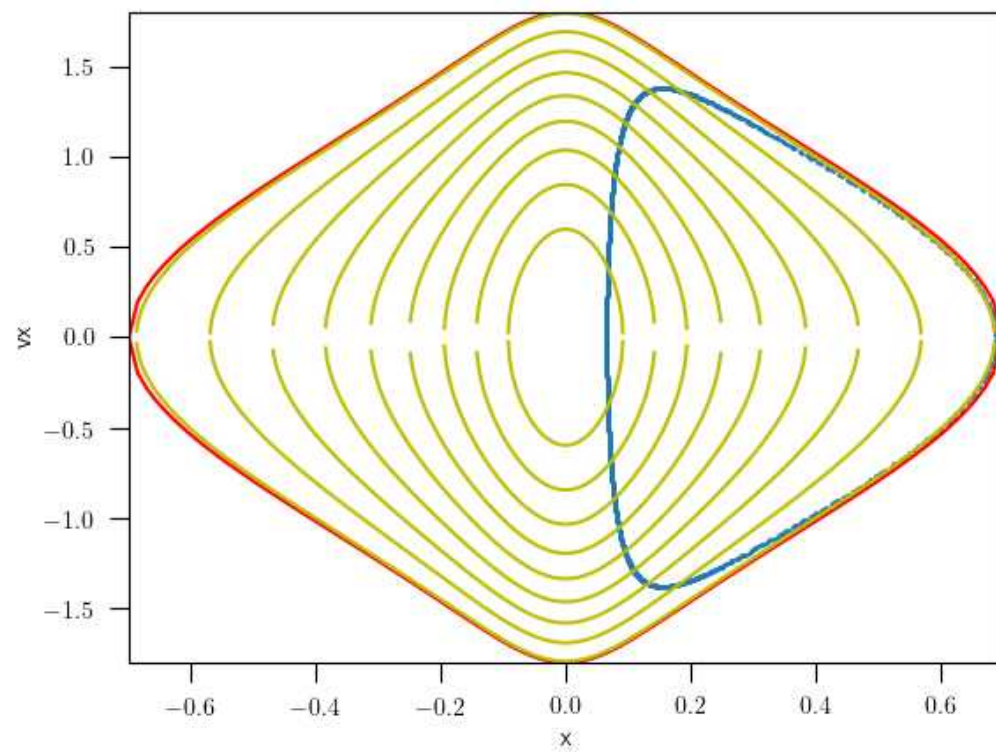
$$\dot{x} = \sqrt{2 (E_x - \phi(x, y=0))}$$

$$H_x$$

$$q = 0.9$$



$$q = 1.0$$

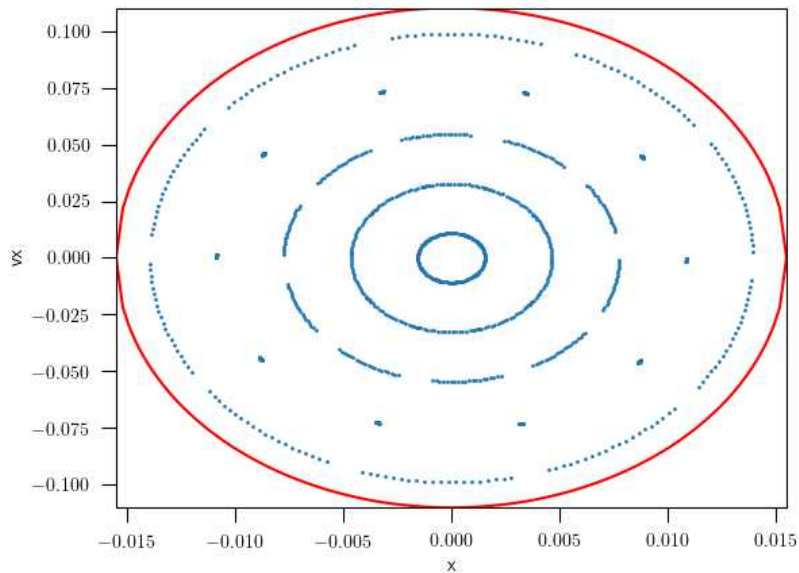


```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --x 0.69 --nlaps 1000 --add_Ix
./mapping.py --V0 1. --Rc 0.14 --q 1.0 -E -0.337 --x 0.69 --nlaps 1000 --add_Ix
```

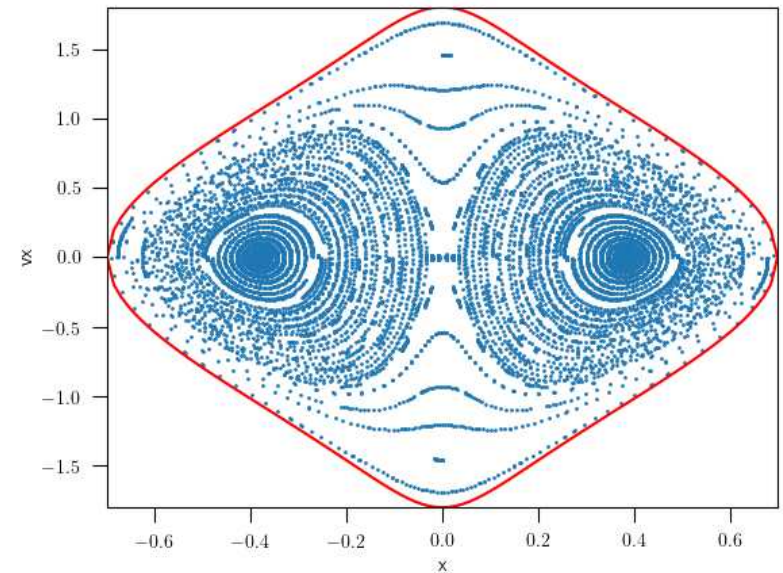
$$R \sim R_c$$

Family decoupling

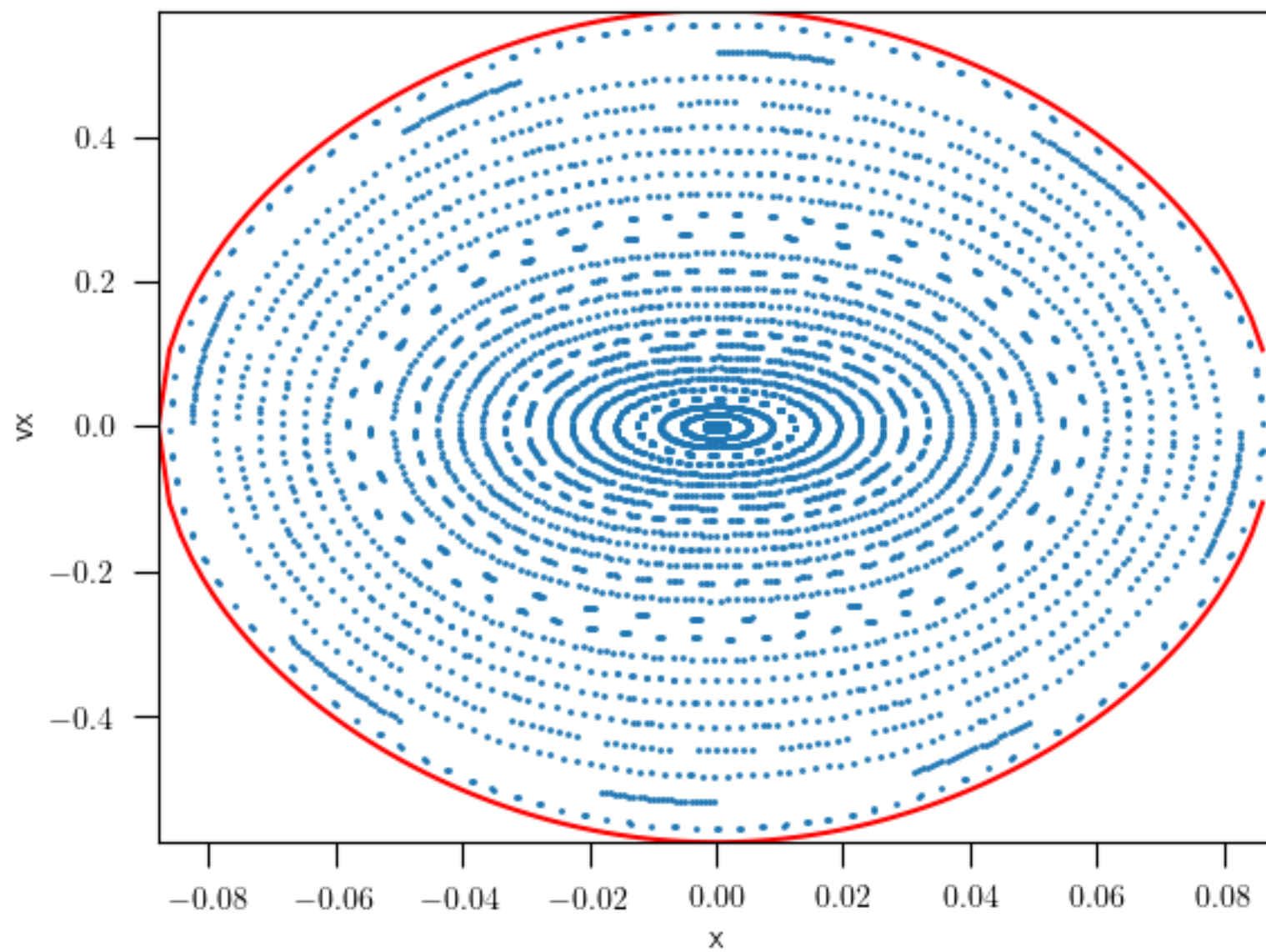
from low energy
1 family



to high energy
2 families

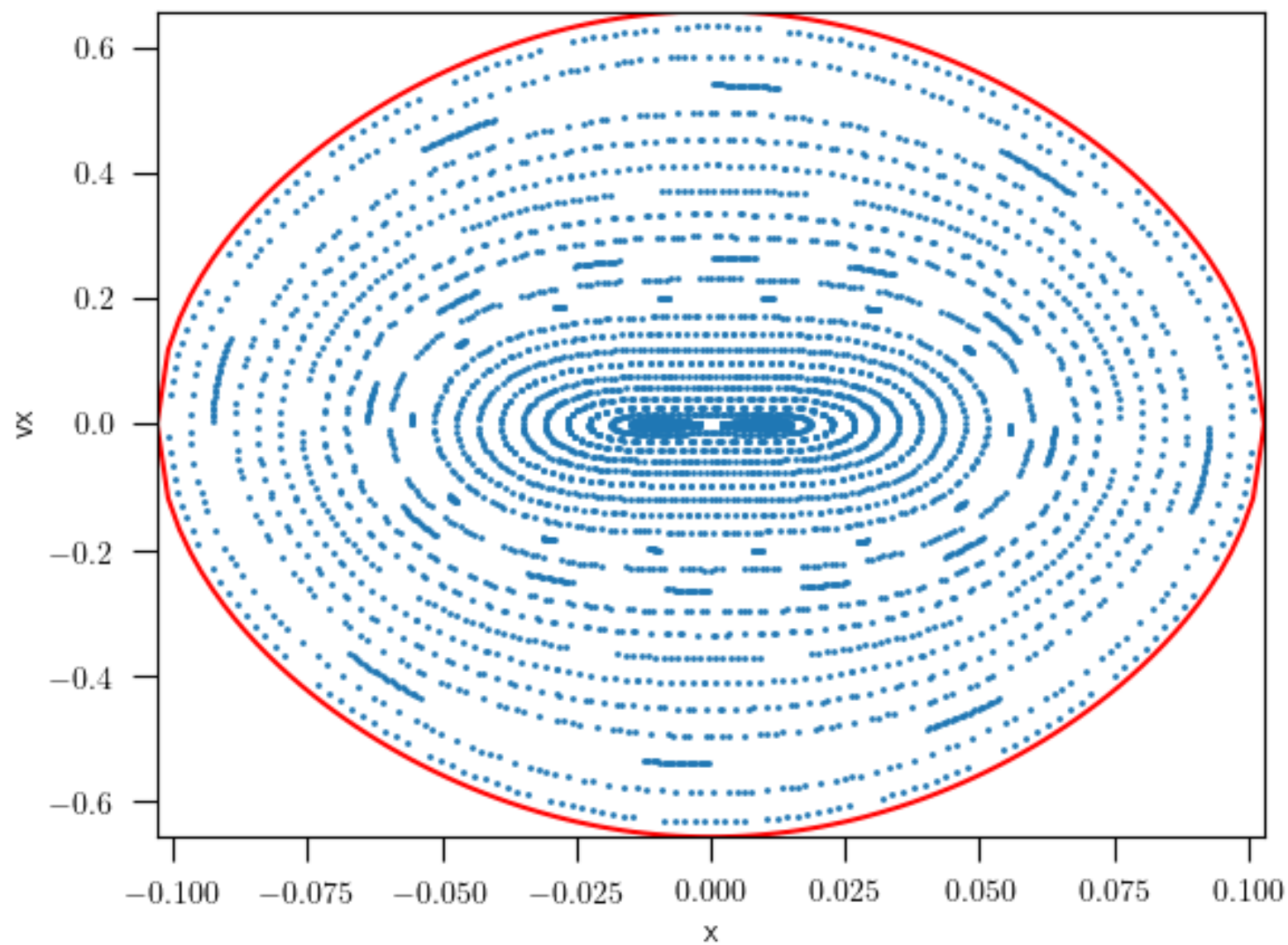


$$E = -1.8$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.80 --norbits 50
```

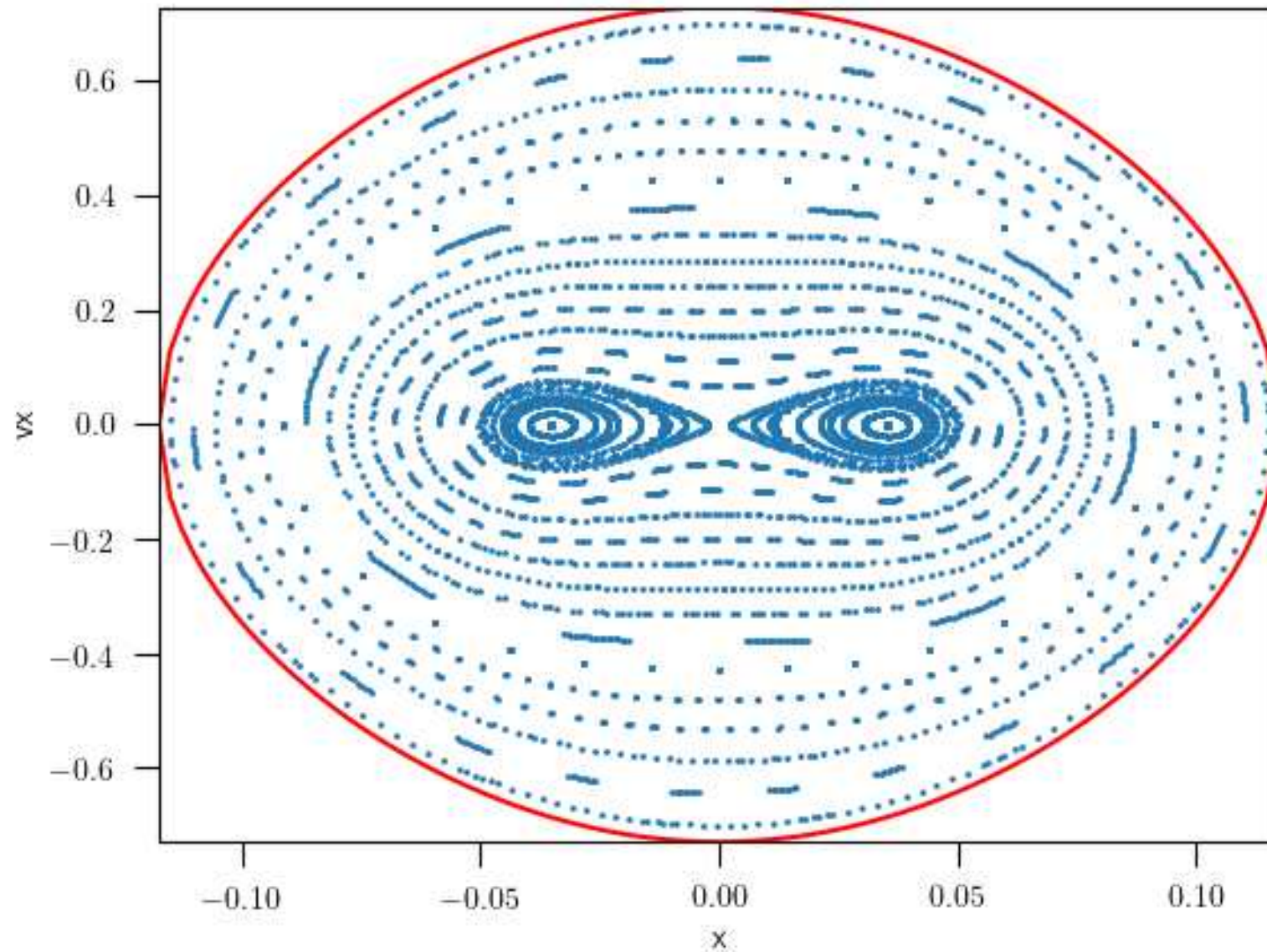
$$E = -1.75$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.75 --norbits 50
```

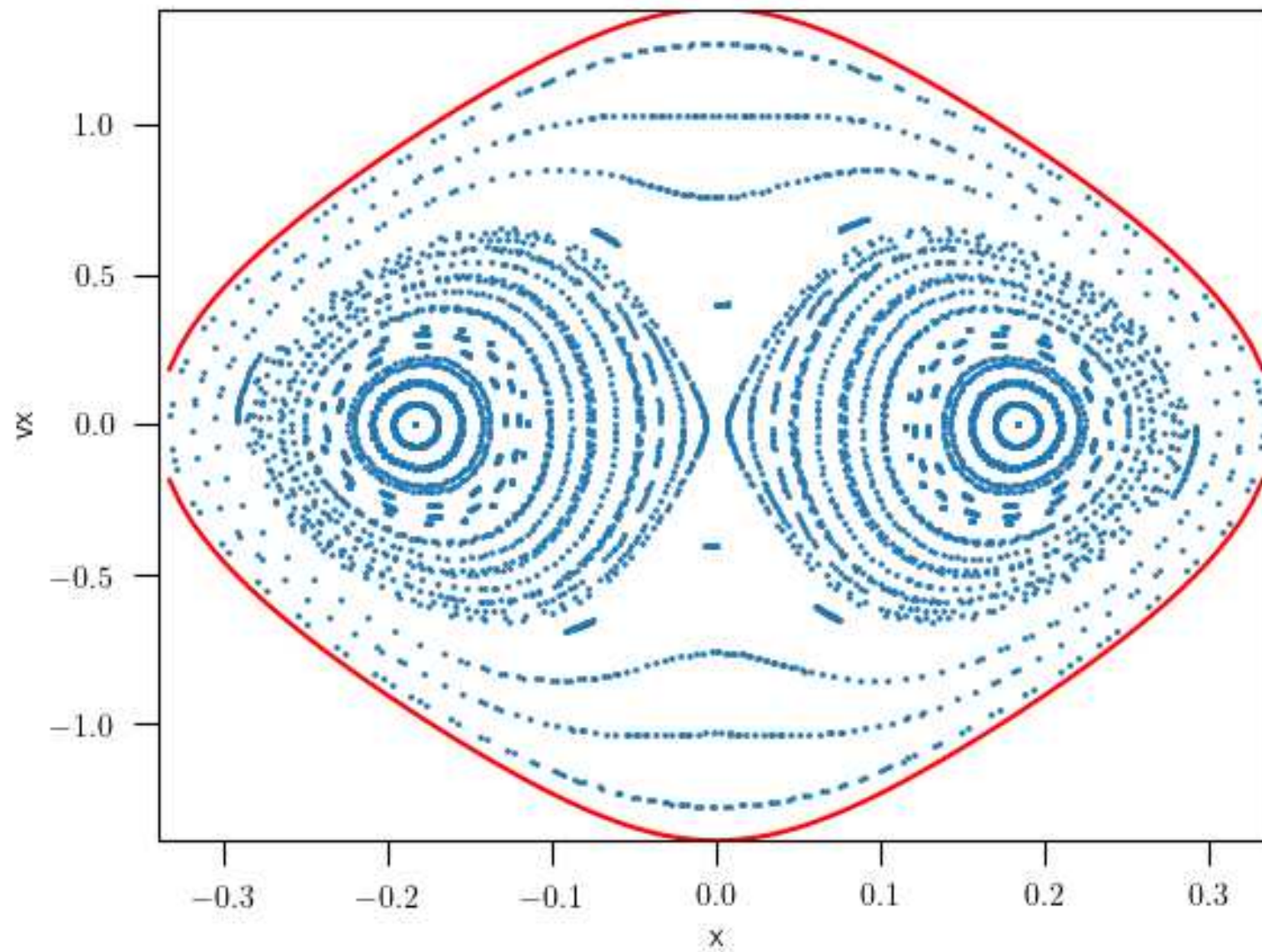
bifurcation

$$E = -1.70$$



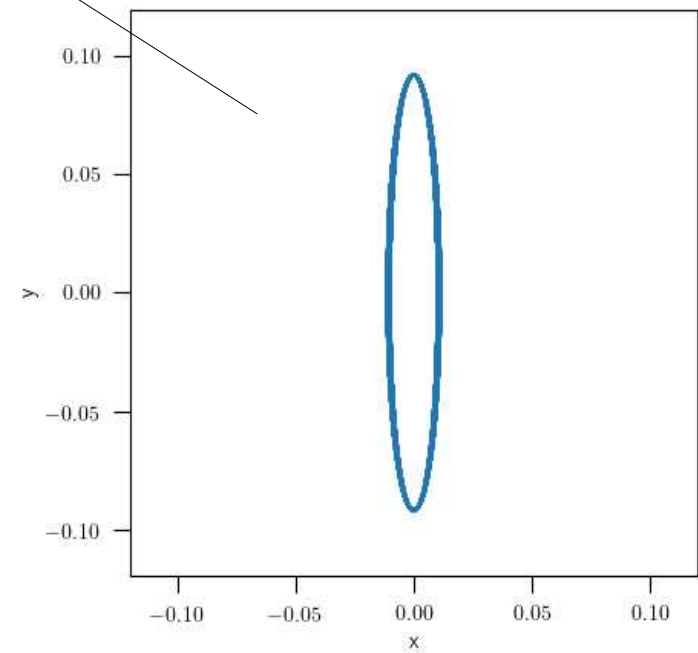
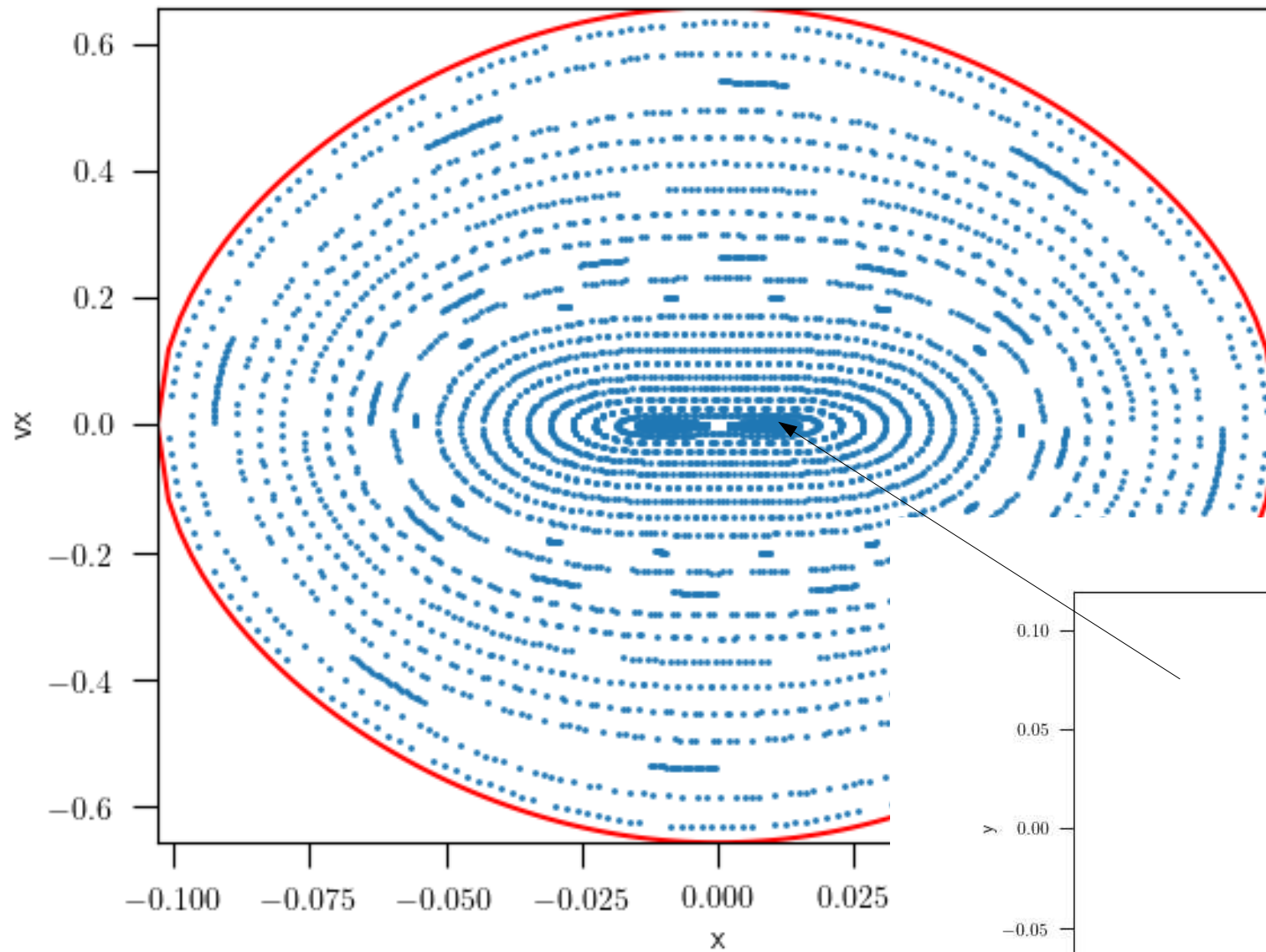
```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.70 --norbits 50
```

$$E = -1.$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1 --norbits 50
```

$$E = -1.75$$

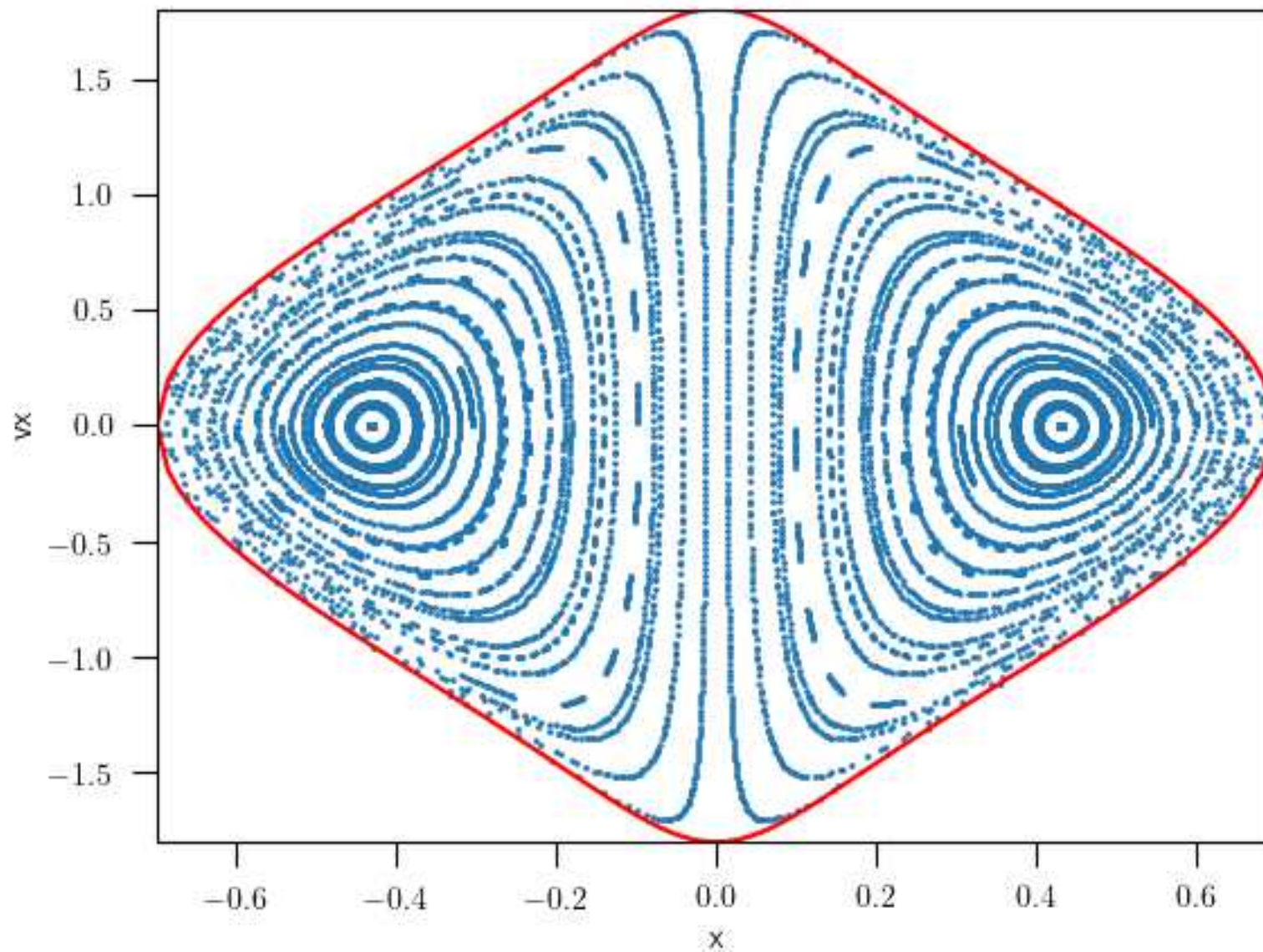


```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.75 --norbits 50
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -1.75 --norbits 18 --x 0.01
```

Evolution with the flattening

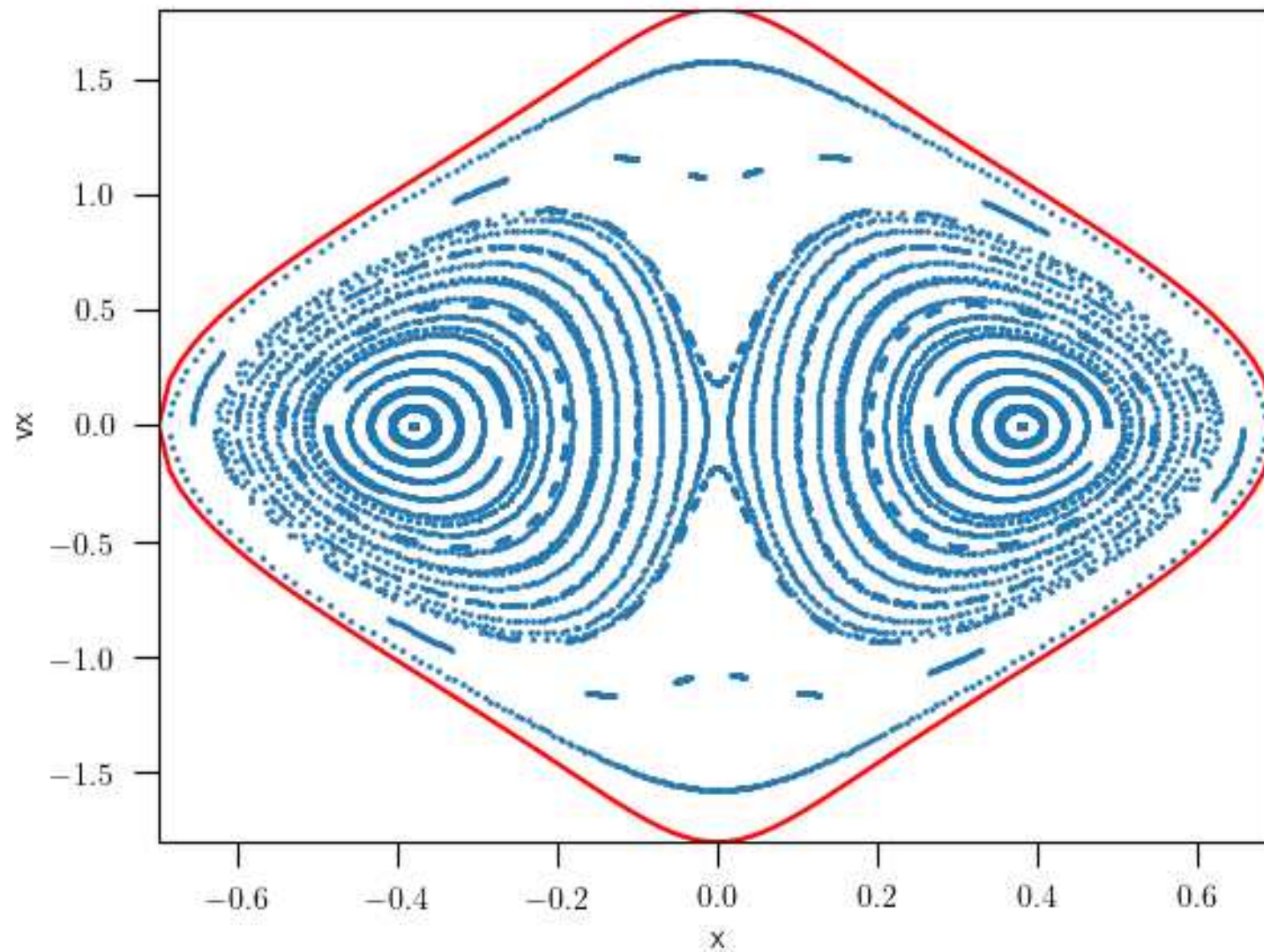
keeping the energy fixed

$$q = 1.0$$



```
./mapping.py --V0 1. --Rc 0.14 --q 1.0 -E -0.337 --norbits 50 --nlaps 200
```

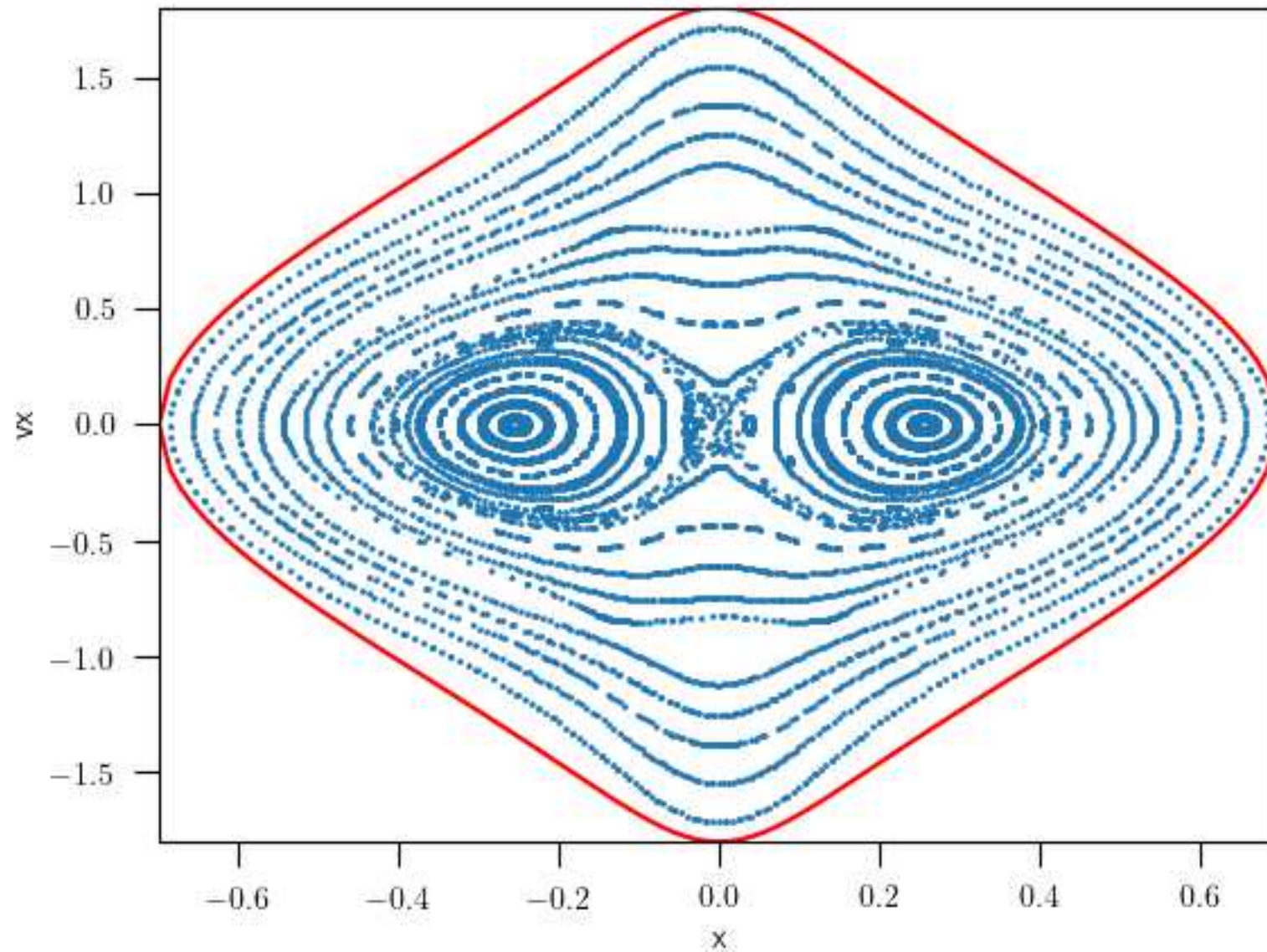
$$q = 0.9$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9 -E -0.337 --norbits 50 --nlaps 200
```

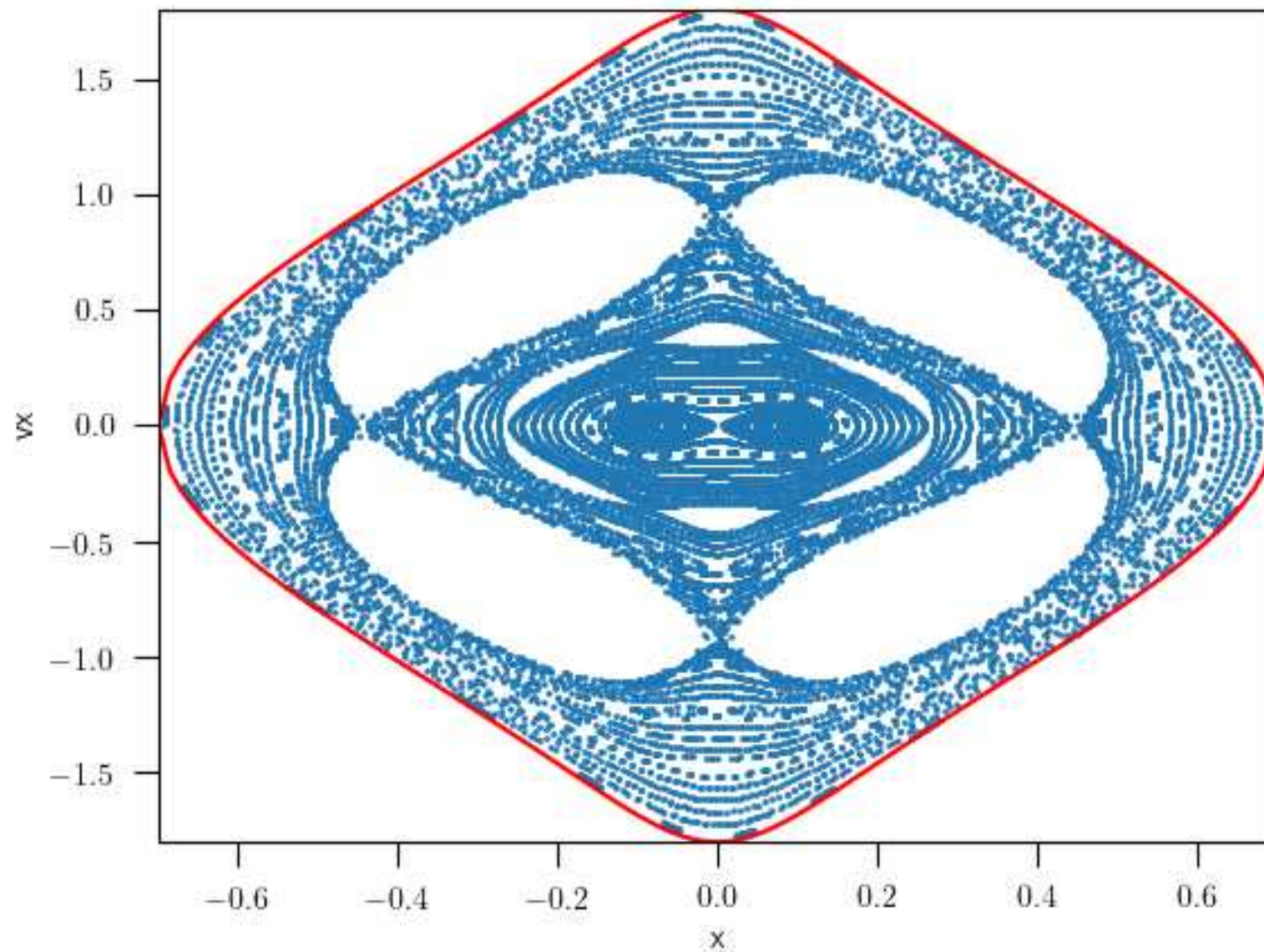
$$q = 0.7$$

Box orbits dominate the
phase space !



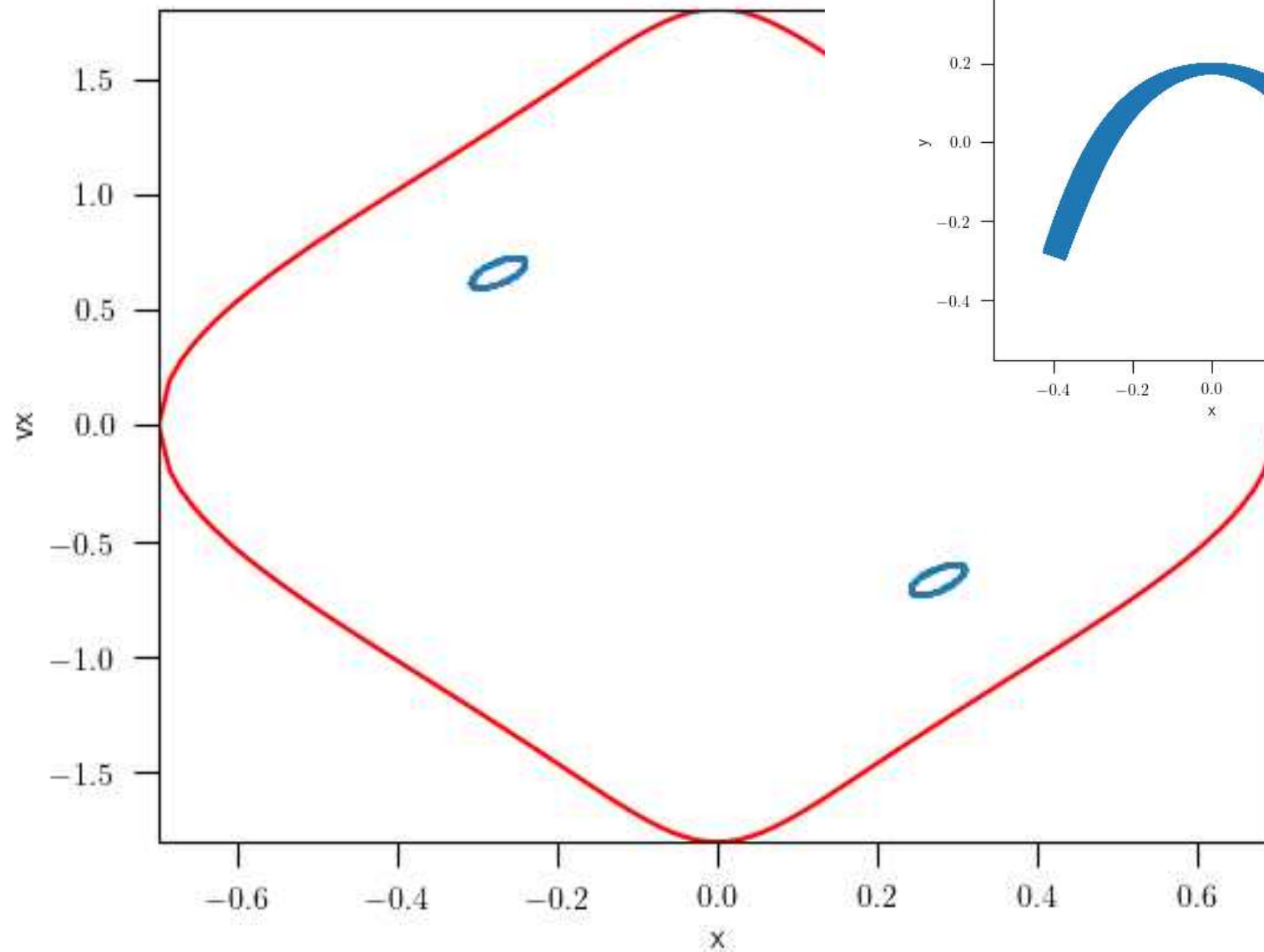
```
./mapping.py --V0 1. --Rc 0.14 --q 0.7 -E -0.337 --norbits 50 --nlaps 200
```

$$q = 0.5$$



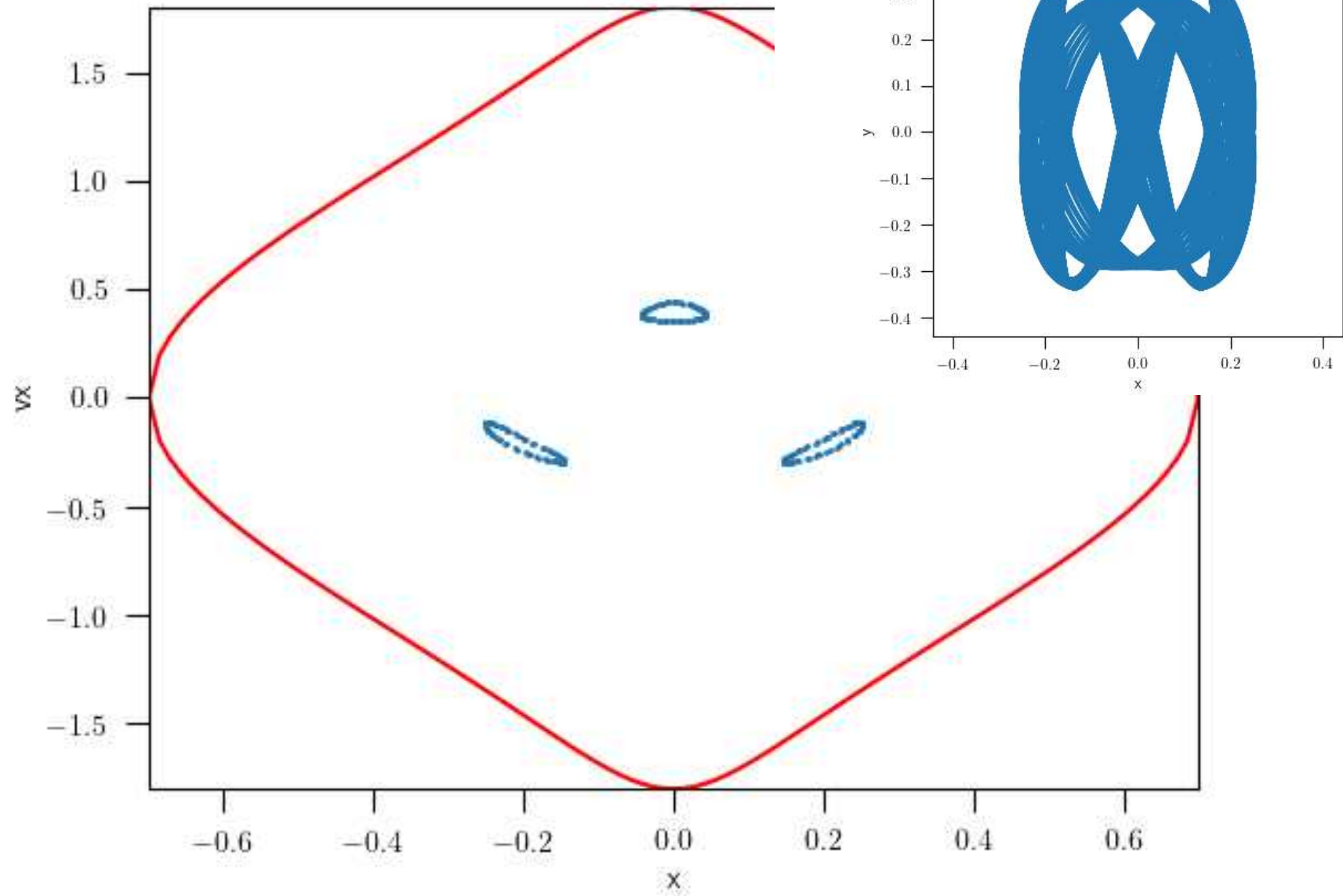
```
./mapping.py --V0 1. --Rc 0.14 --q 0.5 -E -0.337 --norbits 100 --nlaps 200
```

$$q = 0.5$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.5 -E -0.337 --norbits 50 --nlaps 200 --x 0.3 --vx -0.6
```

$$q = 0.5$$



```
./mapping.py --V0 1. --Rc 0.14 --q 0.5 -E -0.337 --norbits 50 --nlaps 200 --x 0.3 --vx -0.6
```

Conclusions

Many 2D bared potential have orbital structures like the logarithmic potential:

- Most orbits respect a 2nd integral (L_z or H_x)
- 2 types of orbits:
 - **Loop** :
 - fixed sense of rotation
 - never reach the centre
 - **Box** :
 - no fixed sense of rotation
 - many reach the centre

Loop orbits dominate when the axis ratio of of the potential is nearly unity.
Box orbits dominate instead.

Stellar Orbits

**Orbits
in planar non-axisymmetric
rotating potentials**

Two dimensional rotating potential



$$\phi(\theta, t) \quad \left\{ \begin{array}{l} \theta \rightarrow L_z \neq \text{cte} \\ t \rightarrow E \neq \text{cte} \end{array} \right.$$

Assume a static rotation of the bar at constant angular frequency Ω_b



Idea : Describe the motion from the rotating frame where the bar is static

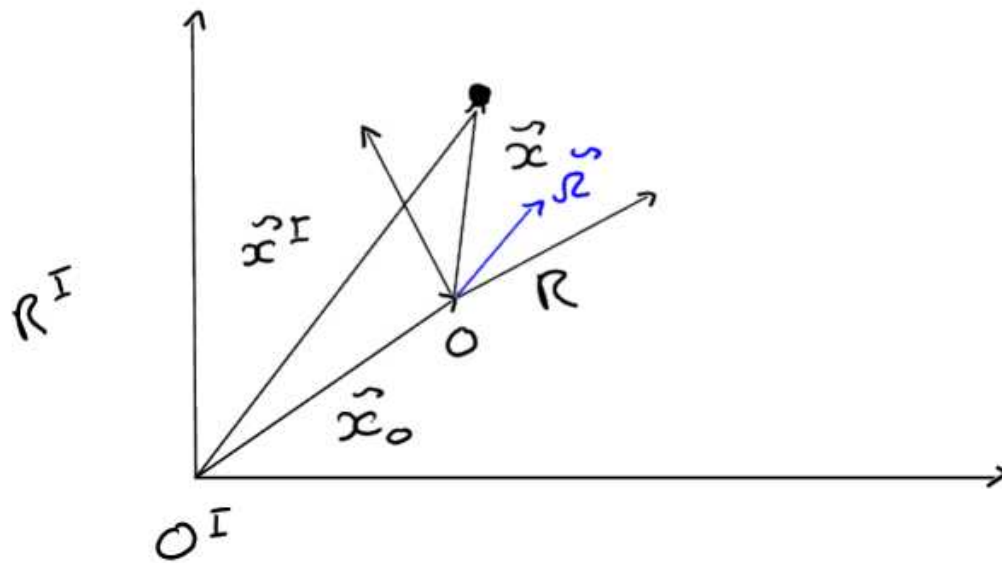
$$(\vec{x}^I, \dot{\vec{x}}^I) \rightarrow (\vec{x}, \dot{\vec{x}})$$

inertial
frame
 R^I

rotating
frame
 R

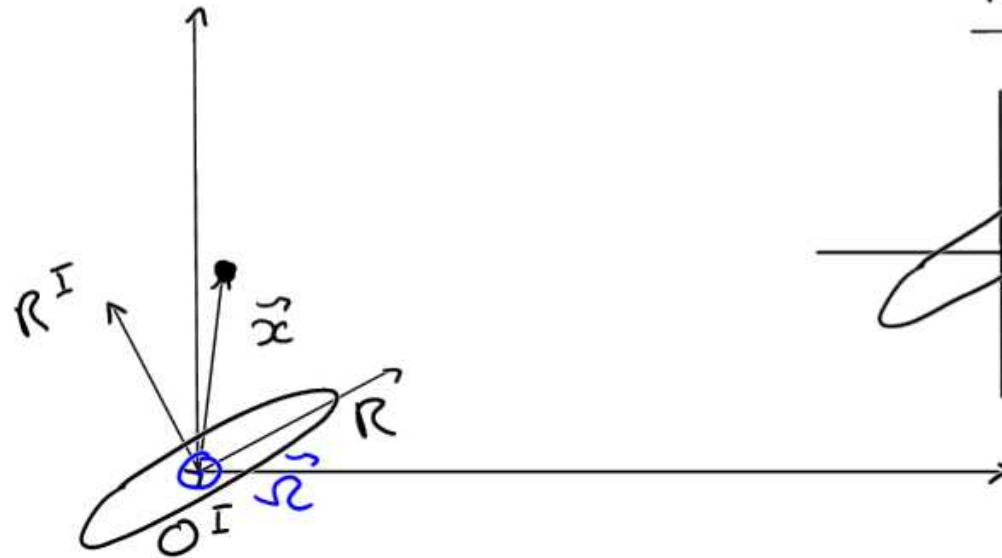
accelerated

Velocity in an accelerated rest frame

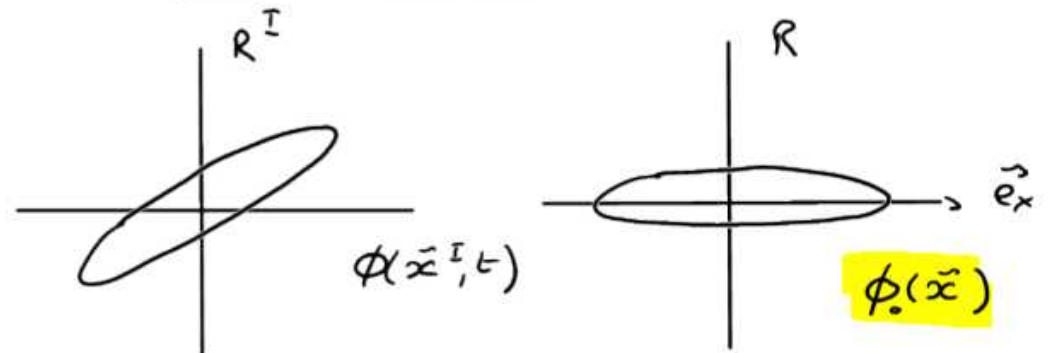


$$\left. \dot{\vec{x}}^I \right|_{R^I} = \left. \dot{\vec{x}}_0^I \right|_{R^I} + \left. \dot{\vec{x}} \right|_R + \vec{\Omega} \times \vec{x} \Big|_R$$

Velocity in a rotating rest frame



R: the bar is aligned with $\vec{e}_x$



$$\left. \dot{\tilde{x}}^I \right|_{R_I} = \cancel{\left. \dot{\tilde{x}}_O^I \right|_{R_I}} + \left. \dot{\tilde{x}} \right|_R + \left. \vec{\Omega} \times \tilde{x} \right|_R$$

$$\tilde{x}_O = 0$$

Lagrangian

In the inertial frame R^I

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} \dot{\vec{x}}^I{}^2 - \phi^I(\vec{x}^I, t)$$

In the rotating frame R

- $\frac{1}{2} \dot{\vec{x}}^I{}^2 \rightarrow \frac{1}{2} (\dot{\vec{x}} + \vec{\Omega}_0 \times \vec{x})^2$
- $\phi^I(\vec{x}^I, t) \rightarrow \phi_0(\vec{x})$

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} (\dot{\vec{x}} + \vec{\Omega}_s \times \vec{x})^2 - \phi_0(\vec{x})$$

Hamiltonian

$$H_J(\vec{x}, \vec{p}) = \frac{1}{2} \vec{p}^2 - \vec{\Omega} \cdot (\vec{p} \times \vec{x}) + \phi(\vec{x})$$

EXERCICE

H_J has no explicit time dependency

$$\Rightarrow H_J = E_J = \text{cte}$$

Jacobi: integral

Equations of motion from Hamilton's equations

$$H_3 = \frac{1}{2} \tilde{\mathbf{p}}^2 - \tilde{\mathbf{Q}} \cdot (\tilde{\mathbf{p}} \times \tilde{\mathbf{x}}) + \phi(\tilde{\mathbf{x}})$$

$$\dot{\tilde{\mathbf{x}}} = \frac{\partial H_3}{\partial \tilde{\mathbf{p}}} = \tilde{\mathbf{p}} - \tilde{\mathbf{Q}} \times \tilde{\mathbf{x}}$$

$$\dot{\tilde{\mathbf{p}}} = -\frac{\partial H_3}{\partial \tilde{\mathbf{x}}} = -\tilde{\nabla} \phi - \tilde{\mathbf{Q}} \times \tilde{\mathbf{p}}$$

EXERCICE

Effective potential

split the kinetic term in the Lagrangian

$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} (\dot{\vec{x}} + \vec{\Omega}_S \times \vec{x})^2 - \phi_0(\vec{x})$$

$$= \frac{1}{2} \dot{\vec{x}}^2 + \dot{\vec{x}} (\vec{\Omega}_S \times \vec{x}) - \underbrace{\phi_0(\vec{x}) + \frac{1}{2} (\vec{\Omega}_S \times \vec{x})^2}_{\text{depends only on } \vec{x}}$$

$$\begin{aligned} \phi_{\text{eff}}(\vec{x}) &:= \phi(\vec{x}) - \frac{1}{2} (\vec{\Omega} \times \vec{x})^2 \\ &= \phi(\vec{x}) - \frac{1}{2} \Omega^2 \vec{x}^2 + \frac{1}{2} (\vec{\Omega} \cdot \vec{x})^2 \end{aligned}$$

$\phi_{\text{centr}}(\vec{x})$: repulsive centrifugal potential

Equations of motion from the Euler - Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} - \frac{\partial \mathcal{L}}{\partial \vec{x}} = 0$$

$$\ddot{\vec{x}} = - \vec{\nabla} \phi_{\text{eff}}(\vec{x}) - 2 (\vec{\Omega} \times \dot{\vec{x}})$$

$$\ddot{\vec{x}} = - \vec{\nabla} \phi(\vec{x}) + \underbrace{\Omega^2 \vec{x} - \vec{\Omega}(\vec{\Omega} \cdot \vec{x})}_{\text{centrifugal force}} - \underbrace{2 (\vec{\Omega} \times \dot{\vec{x}})}_{\text{Coriolis force}}$$

centrifugal force

Coriolis force

$$= \Omega^2 \vec{x} \quad \text{if } \vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \quad \text{and } z=0$$

Stationary points

$$\dot{\vec{x}} = \ddot{\vec{x}} = 0 \quad (\text{in the rotating frame})$$

1) The point **co-rotate with the bar** ($V_{\perp} = \Omega R$)

2) with $\ddot{\vec{x}} = -\vec{\nabla}\phi_{\text{eff}} - 2\vec{\Omega} \times \dot{\vec{x}}_{\vec{\omega}=0} = 0$

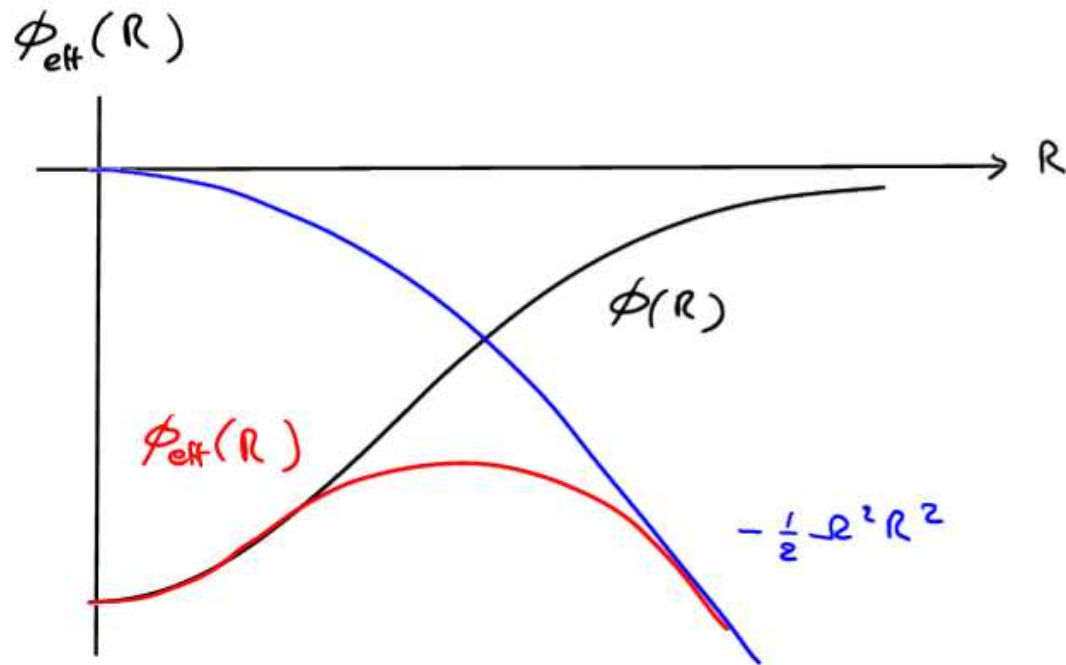
$$\vec{\nabla}\phi_{\text{eff}} = 0$$

"gravity" counter balance
the centrifugal force

Positions of the stationary points

- 1) assume $y = 0$ $\vec{\Omega}_b = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$
- 2) assume an axi-symmetric system

$$\phi_{\text{eff}}(\vec{x}) \rightarrow \phi_{\text{eff}}(R) = \phi(R) - \frac{1}{2} \Omega^2 R^2$$

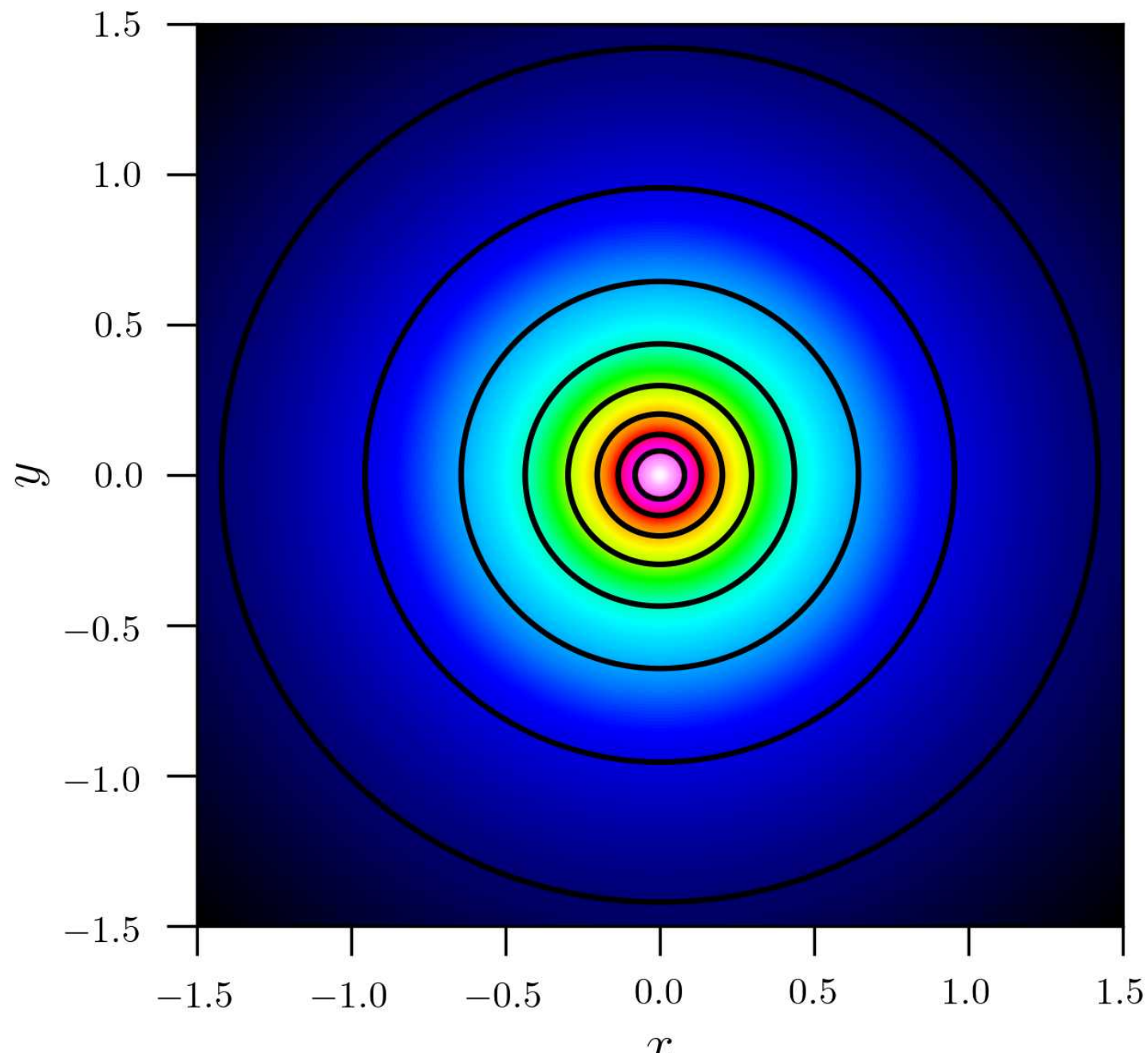


$$\vec{\nabla} \phi_{\text{eff}}(\vec{x}) = 0 \equiv \frac{\partial}{\partial R} \phi_{\text{eff}}(R) = 0$$

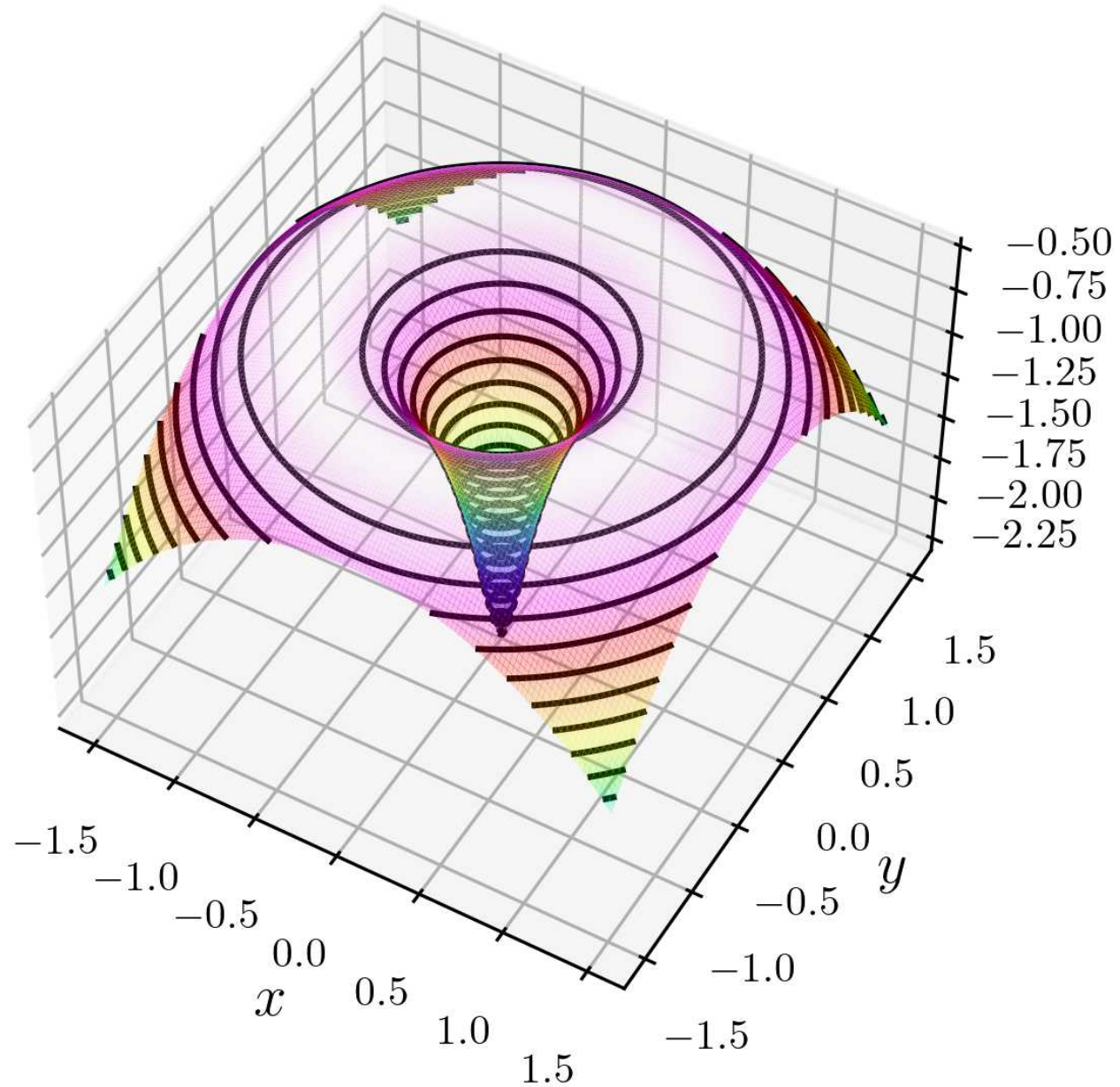
$$\frac{\partial}{\partial R} \phi(R) = \Omega^2 R$$

$\Rightarrow$ selection of a circular orbit in the inertial frame

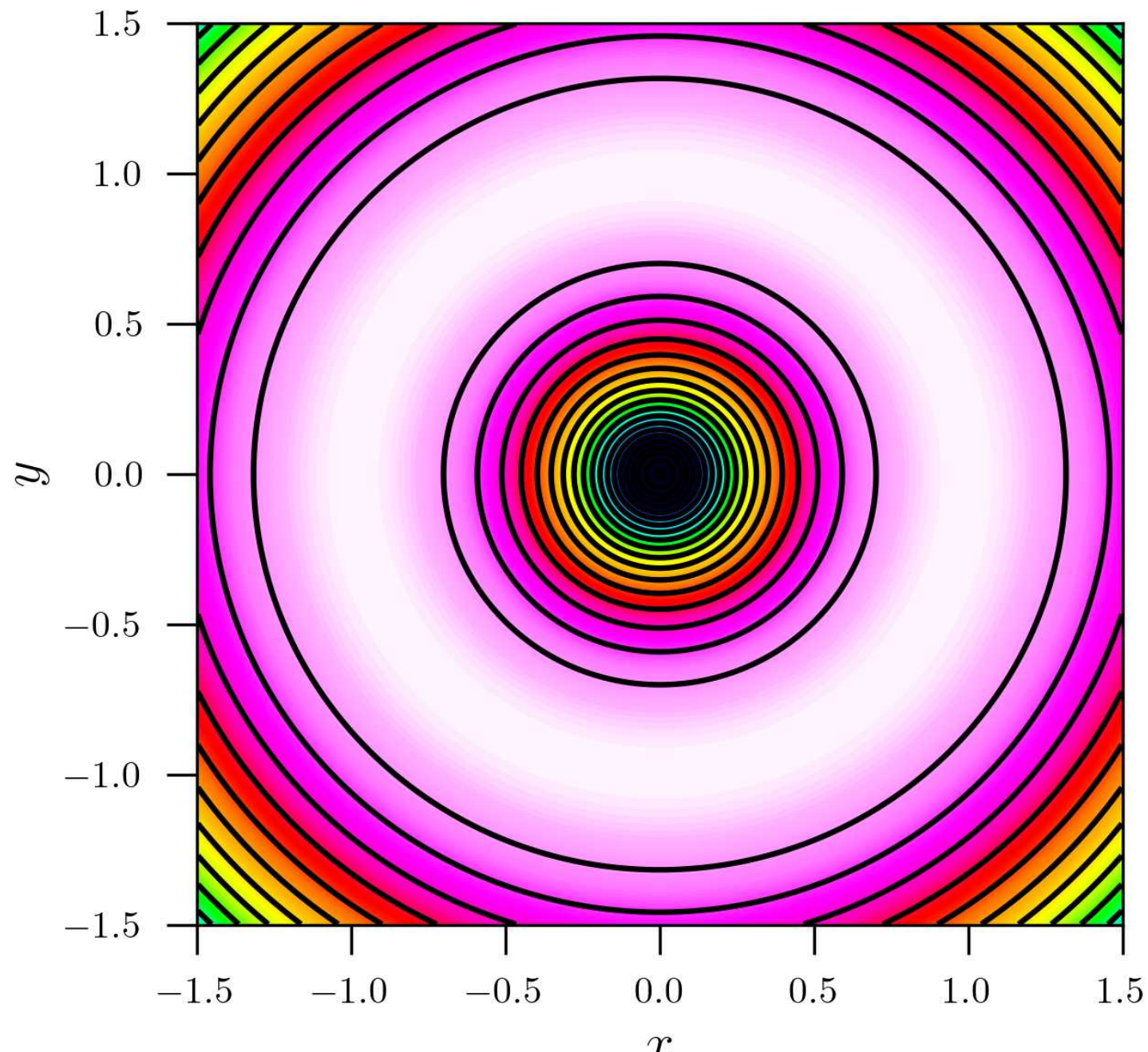
Bar potential (density)
(Logarithmic potential: $V_0=1$, $R_c=0.1$
 $q=1.0$)



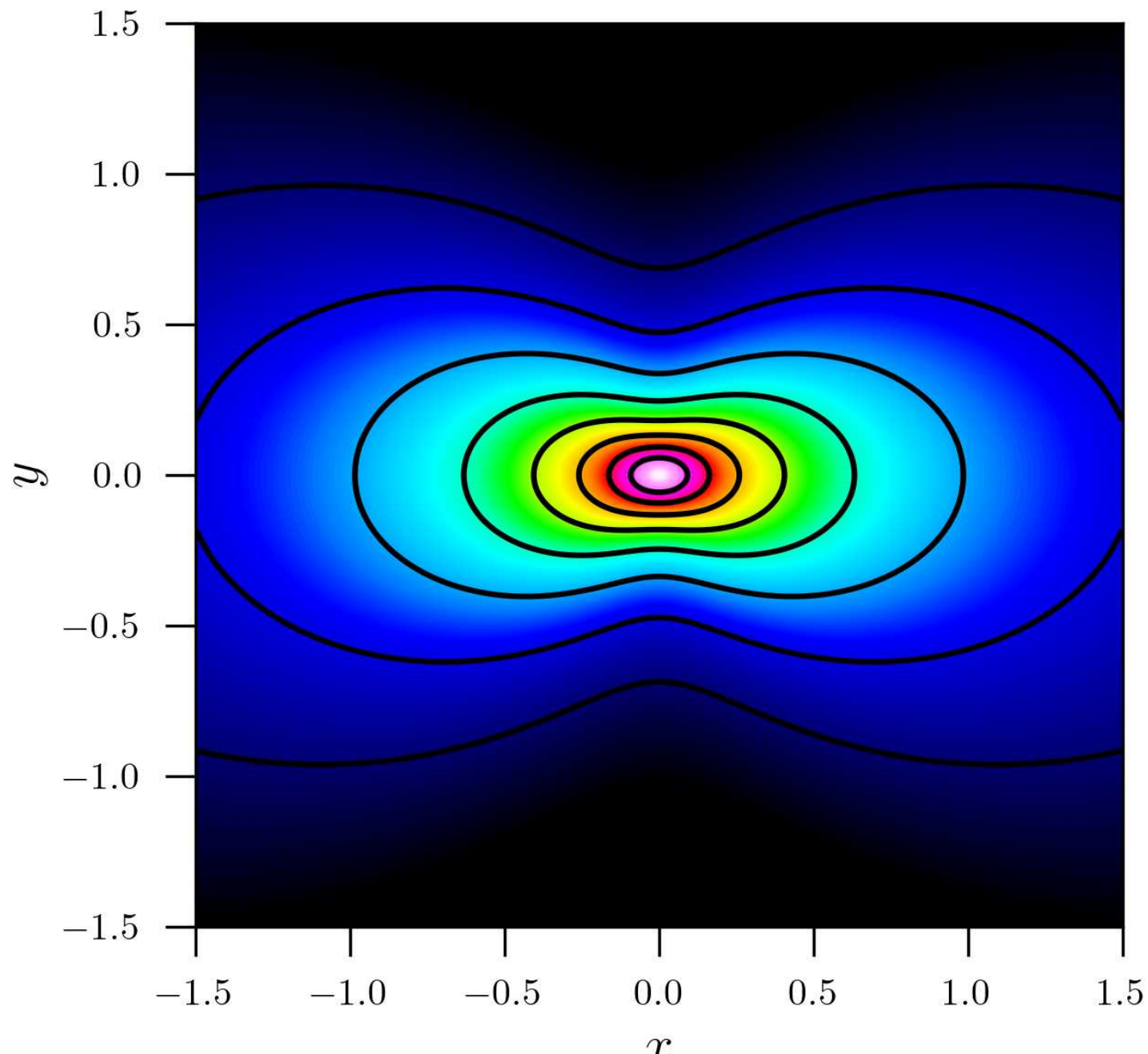
Effective Potential
(Logarithmic potential: $V_0=1$, $R_c=0.1$)
 $q=1.0$
Rotation : $\Omega=1$)



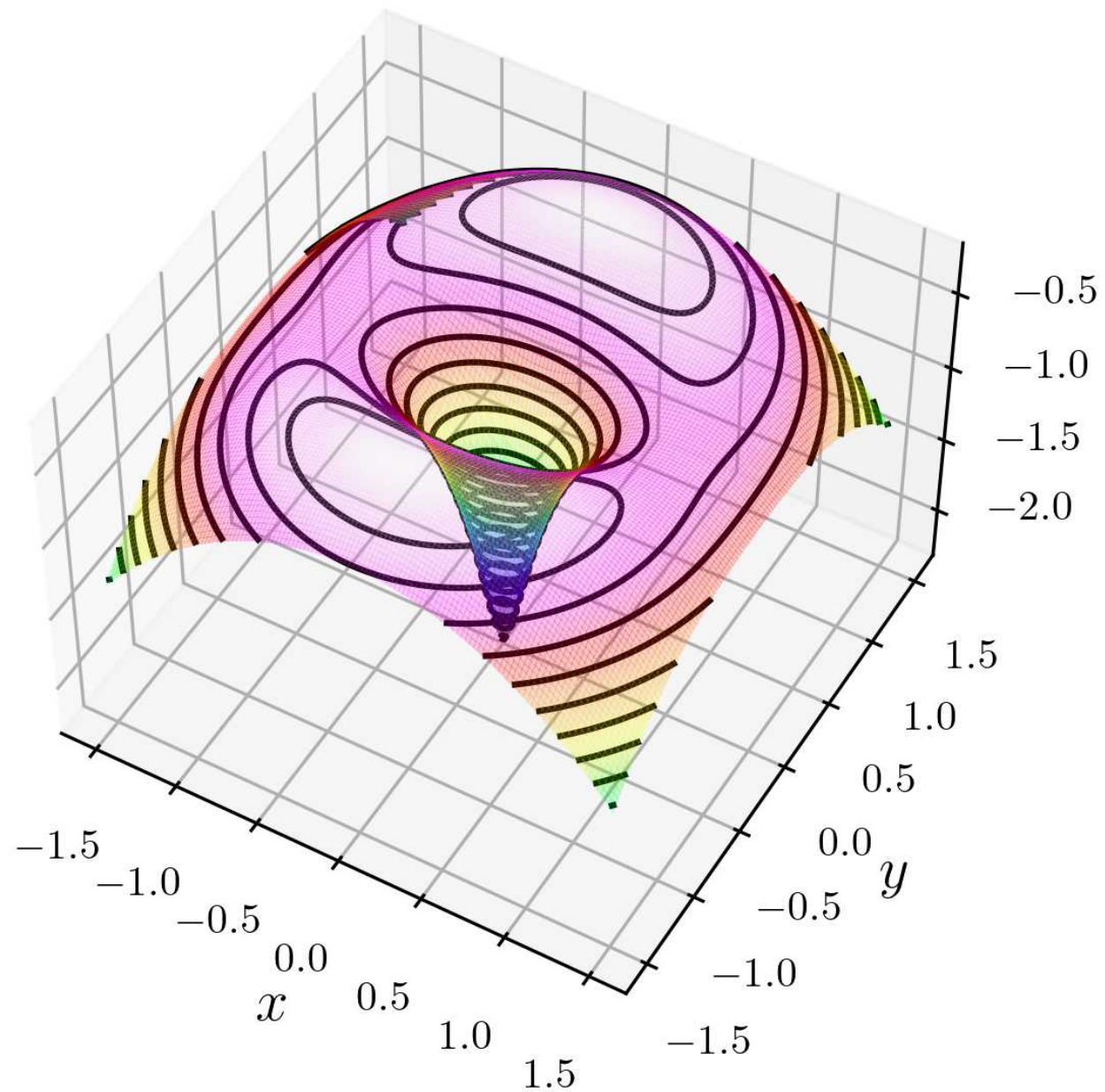
Effective Potential
(Logarithmic potential: $V_0=1$, $R_c=0.1$
 $q=1.0$
Rotation : $\Omega=1$)



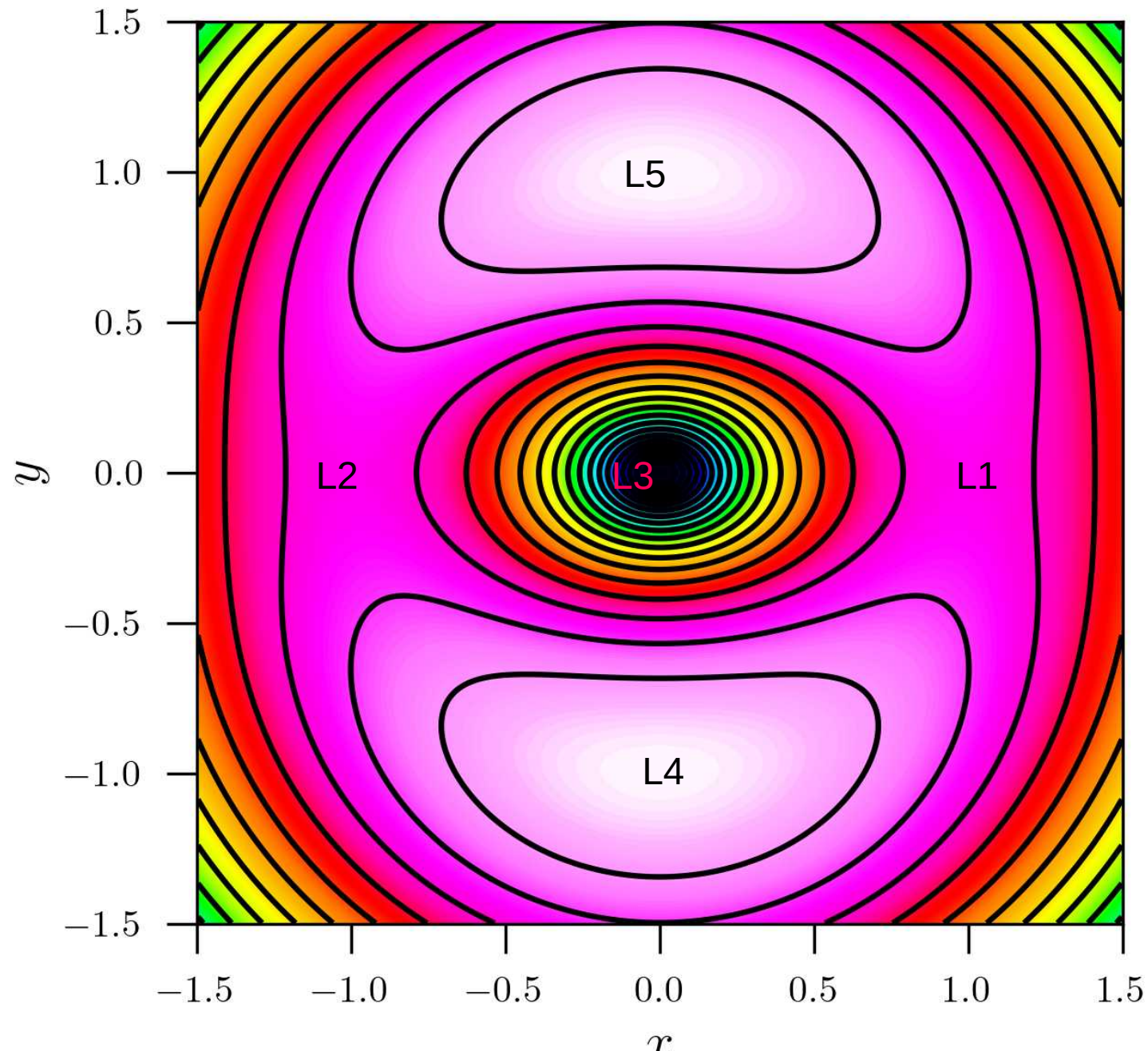
Bar potential
(Logarithmic potential: $V_0=1$, $R_c=0.1$
 $q=0.75$)



Effective Potential
(Logarithmic potential: $V_0=1$, $R_c=0.1$)
 $q=0.75$
Rotation : $\Omega=1$)



Effective Potential
(Logarithmic potential: $V_0=1$, $R_c=0.1$
 $q=0.75$
Rotation : $\Omega=1$)



Stellar Orbits

Orbits around Lagrange points

Stability of orbits around Lagrange points

Expand the effective potential in Taylor series around the Lagrange points (x_L, y_L)

$$\begin{aligned}\phi_{\text{eff}}(x, y) \approx & \phi_{\text{eff}}(x_L, y_L) + \cancel{\frac{\partial \phi_{\text{eff}}}{\partial x}(x - x_L)} + \cancel{\frac{\partial \phi_{\text{eff}}}{\partial y}(y - y_L)} \\ & + \frac{1}{2} \frac{\partial^2 \phi_{\text{eff}}}{\partial x^2} (x - x_L)^2 + \frac{1}{2} \frac{\partial^2 \phi_{\text{eff}}}{\partial y^2} (y - y_L)^2 + \cancel{\frac{1}{2} \frac{\partial^2 \phi_{\text{eff}}}{\partial x \partial y} (x - x_L)(y - y_L)}\end{aligned}$$

by symmetry of the bar, if it is aligned with $\bar{x}$

Now we define

$$\begin{aligned}\xi &:= x - x_L & \phi_{xx} &:= \frac{\partial^2 \phi_{\text{eff}}}{\partial x^2} \\ \eta &:= y - y_L & \phi_{yy} &:= \frac{\partial^2 \phi_{\text{eff}}}{\partial y^2}\end{aligned}$$

$$\phi_{\text{eff}}(\xi, \eta) = \phi_{\text{eff}}(0, 0) + \frac{1}{2} \phi_{xx} \xi^2 + \frac{1}{2} \phi_{yy} \eta^2$$

Equations of motions $\ddot{\vec{x}} = - \vec{\nabla} \phi_{\text{eff}} - 2(\vec{\Omega} \times \dot{\vec{x}})$

in the plane $z=0$ assuming $\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

$$\begin{cases} \ddot{x} = - \frac{\partial \phi_{\text{eff}}}{\partial x} + 2\Omega \dot{y} \\ \ddot{y} = - \frac{\partial \phi_{\text{eff}}}{\partial y} - 2\Omega \dot{x} \end{cases}$$

$$\begin{cases} \ddot{\xi} = + 2\Omega \dot{\eta} - \phi_{xx} \xi \\ \ddot{\eta} = - 2\Omega \dot{\xi} - \phi_{yy} \eta \end{cases}$$

We assume solutions of the form

$$\begin{cases} \xi(t) = X e^{\lambda t} \\ \eta(t) = Y e^{\lambda t} \end{cases} \quad X, Y, \lambda \in \mathbb{C}$$

The EoM become

$$\begin{cases} (\lambda^2 + \phi_{xx}) X - (2\lambda\Omega) Y = 0 \\ (2\lambda\Omega) X + (\lambda^2 + \phi_{yy}) Y = 0 \end{cases}$$

$$\begin{pmatrix} \lambda^2 + \phi_{xx} & -2\lambda\Omega \\ 2\lambda\Omega & \lambda^2 + \phi_{yy} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

M

simple linear
equation

Non trivial solutions (i.e. $X \neq 0$, $Y \neq 0$) only if $\text{Det}(M) = 0$

$$\text{Det } M = \lambda^4 + \lambda^2 (\phi_{xx} + \phi_{yy} + 4\Omega^2) + \phi_{xx}\phi_{yy} = 0$$

"characteristic equation"

Solutions

(4 roots, two are coupled)

- if λ is a solution $\Rightarrow -\lambda$ is a solution

- if λ is real $\left\{ \begin{array}{l} \xi(t) = X e^{\lambda t} \rightarrow \text{exponential growth} \\ \eta(t) = Y e^{\lambda t} \rightarrow \text{exponential growth} \end{array} \right.$

$\rightarrow$ the star leaves the Lagrange point

UNSTABLE

- if all λ are purely complex $\lambda_1 = \alpha i$ $\lambda_2 = -\alpha i$ $\lambda_3 = \beta i$ $\lambda_4 = -\beta i$ $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} \xi(t) &= \operatorname{Re} \left(X_1' e^{i\alpha t} + X_2' e^{-i\alpha t} + X_3' e^{i\beta t} + X_4' e^{-i\beta t} \right) \\ &= X_1' \cos(\alpha t) + X_2' \cos(-\alpha t) + X_3' \cos(\beta t) + X_4' \cos(-\beta t) \\ &= X_1' \cos(\alpha t) + X_2' \cos(\beta t) \end{aligned}$$

idem for $\eta(t)$, so we get

$$\begin{cases} \xi(t) = X_1 \cos(\alpha t) + X_2 \cos(\beta t) \\ \eta(t) = Y_1 \cos(\alpha t) + Y_2 \cos(\beta t) \end{cases}$$

STABLE

$$\text{with } \begin{cases} Y_1 = \frac{\phi_{xx} - \alpha^2}{2\Omega\alpha} X_1 = \frac{2\Omega\alpha}{\phi_{yy} - \alpha^2} X_1 \\ Y_2 = \frac{\phi_{xx} - \beta^2}{2\Omega\beta} X_1 = \frac{2\Omega\beta}{\phi_{yy} - \beta^2} X_2 \end{cases}$$

It is possible to demonstrate that :

• At L_3 i.e. $\min(\phi_{\text{eff}})$

always stable

• At L_2, L_3 i.e. the saddles points

always unstable

• At L_4, L_5 i.e. $\max(\phi_{\text{eff}})$

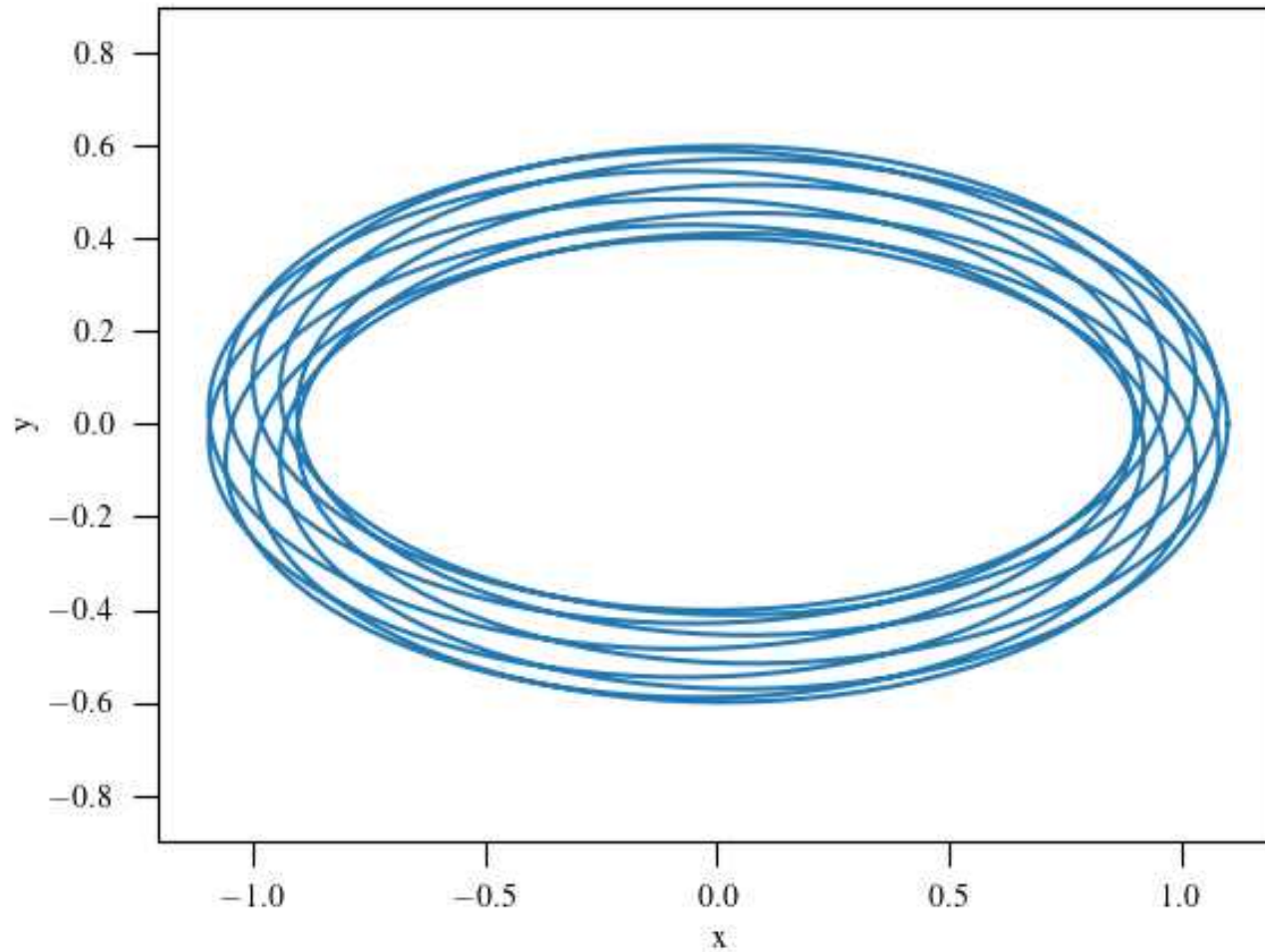
stable or unstable



depends on the detail of
the potential

Note:

The stability comes from
the Coriolis force (see Padmanabhan)



always stable

always unstable

stable or unstable

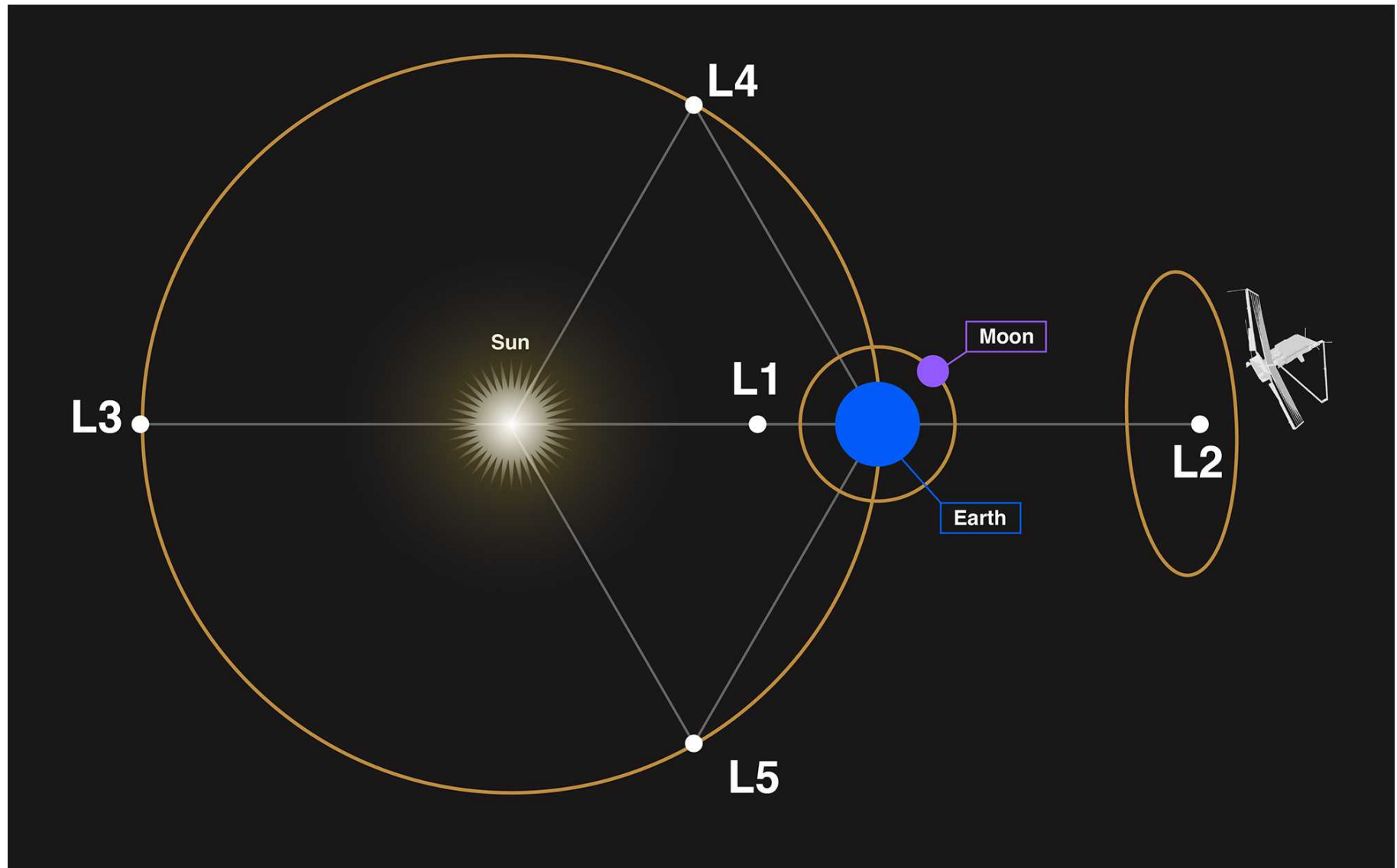


the detail of
the potential

Note:

The stability comes from
the Coriolis force (see Padmanabhan)

Lagrange points in celestial mechanics



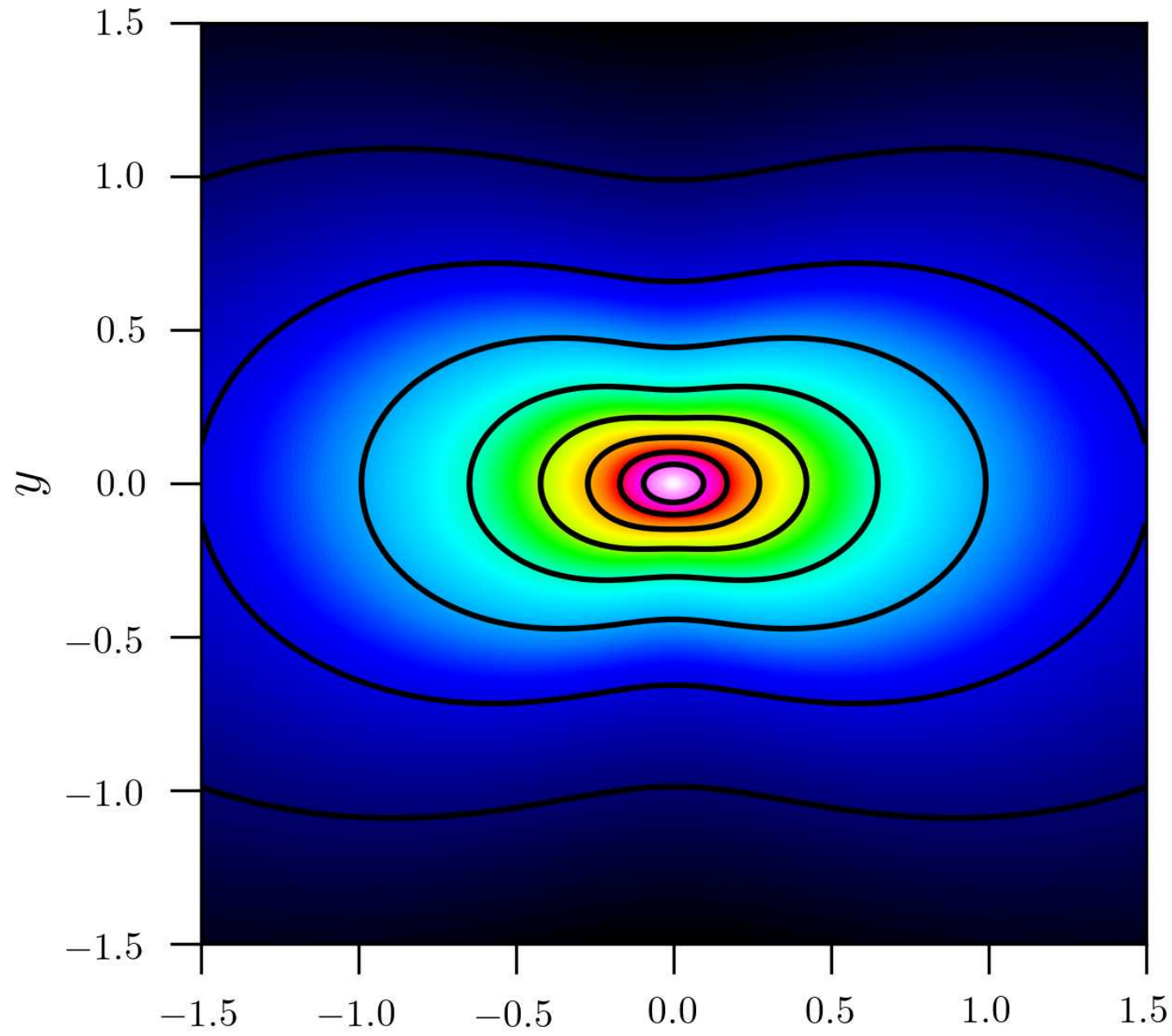
Stellar Orbits

**Orbits not confined to
Lagrange points**

Bar model : Logarithmic potential:
Vo=01 Rc=0.1 q=0.8)

$$\Phi_{\log}(x, y) = \frac{1}{2} V_0^2 \ln \left(R_c^2 + x^2 + \left(\frac{y}{q} \right)^2 \right)$$

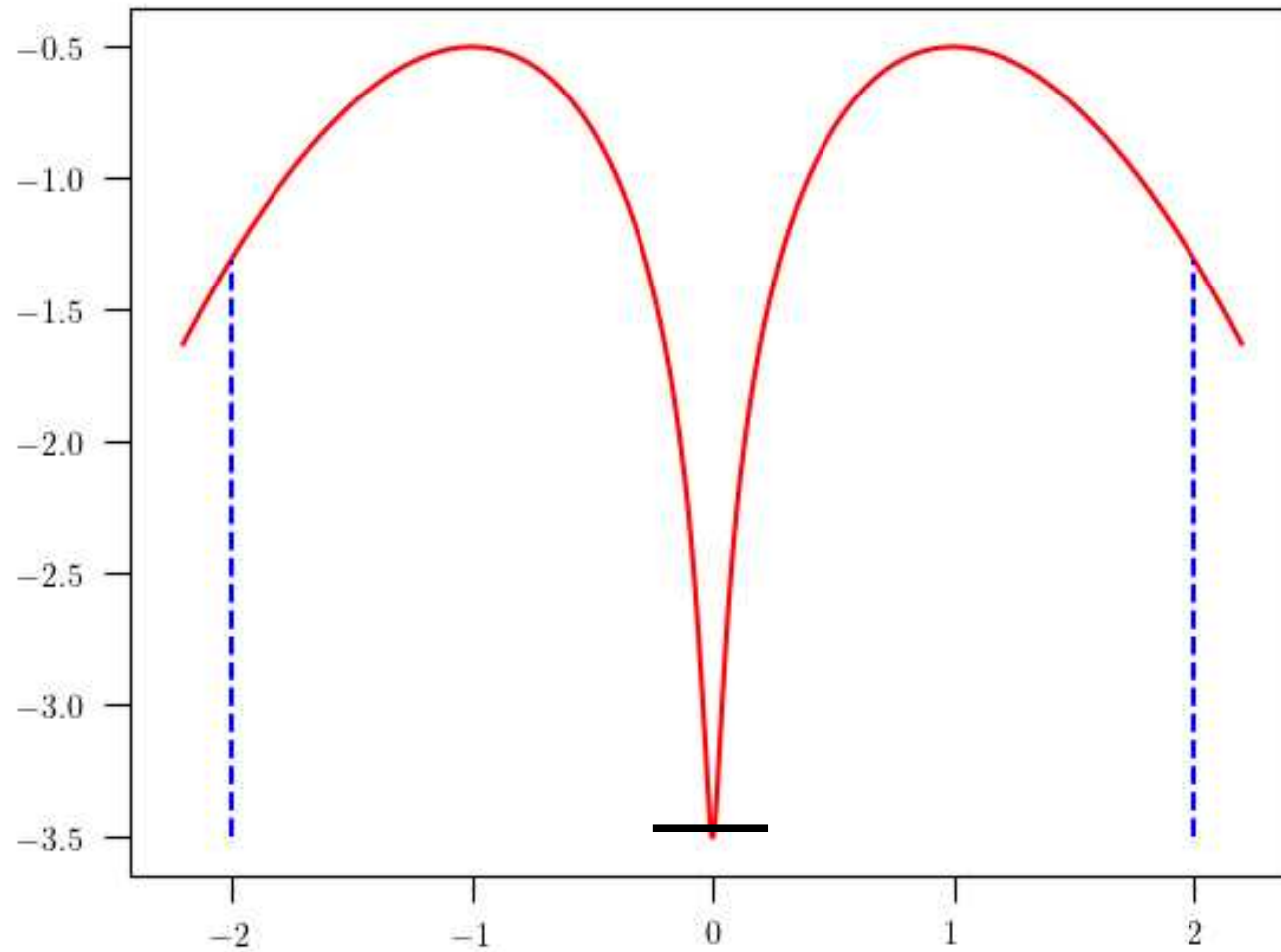
$$\Omega_p \neq 0$$



Low energy orbits

$$R \ll R_{\text{corot}}$$

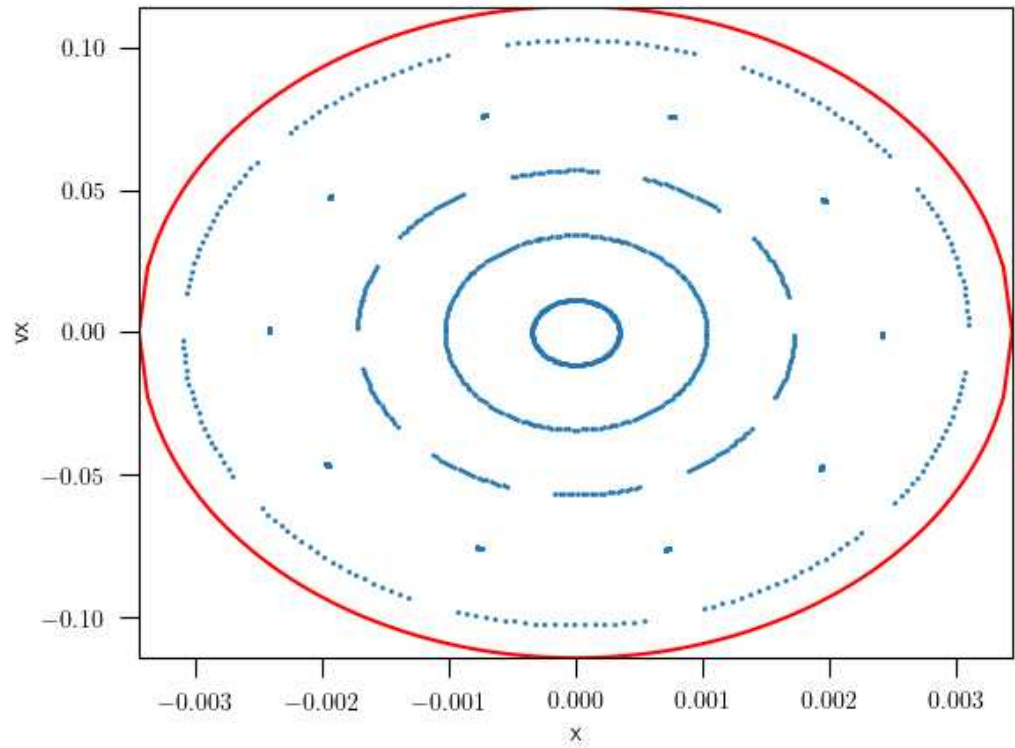
Potential and energy



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

Orbits around L_3

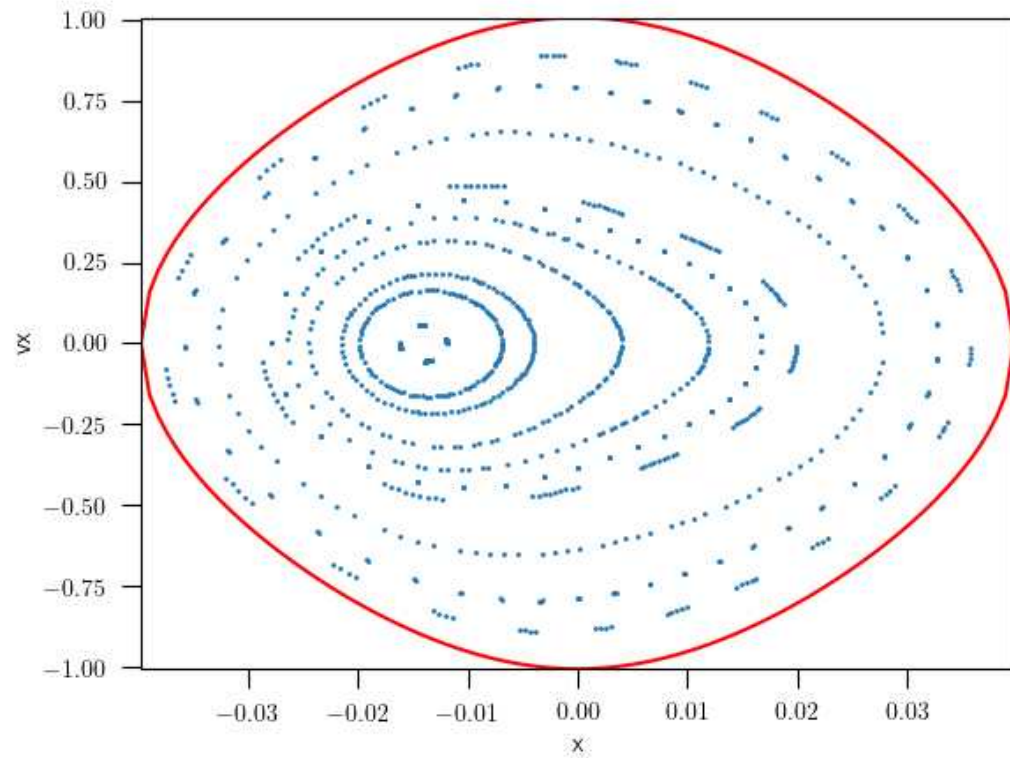
$$\Omega = 0$$



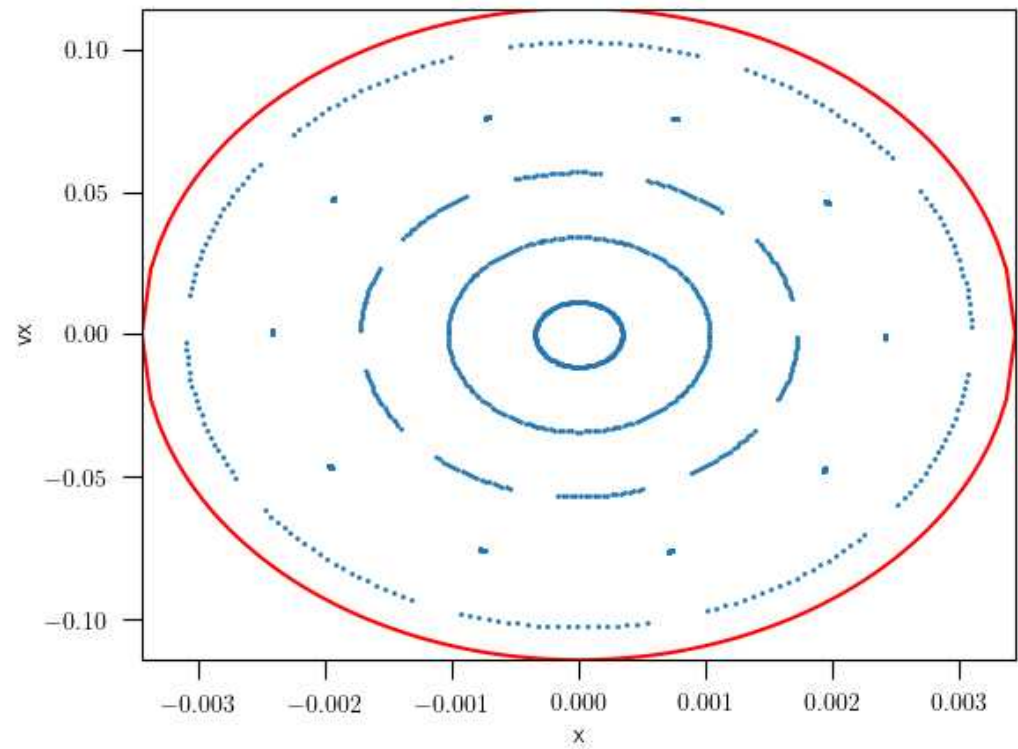
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Orbits around L_3

$$\Omega = 1$$



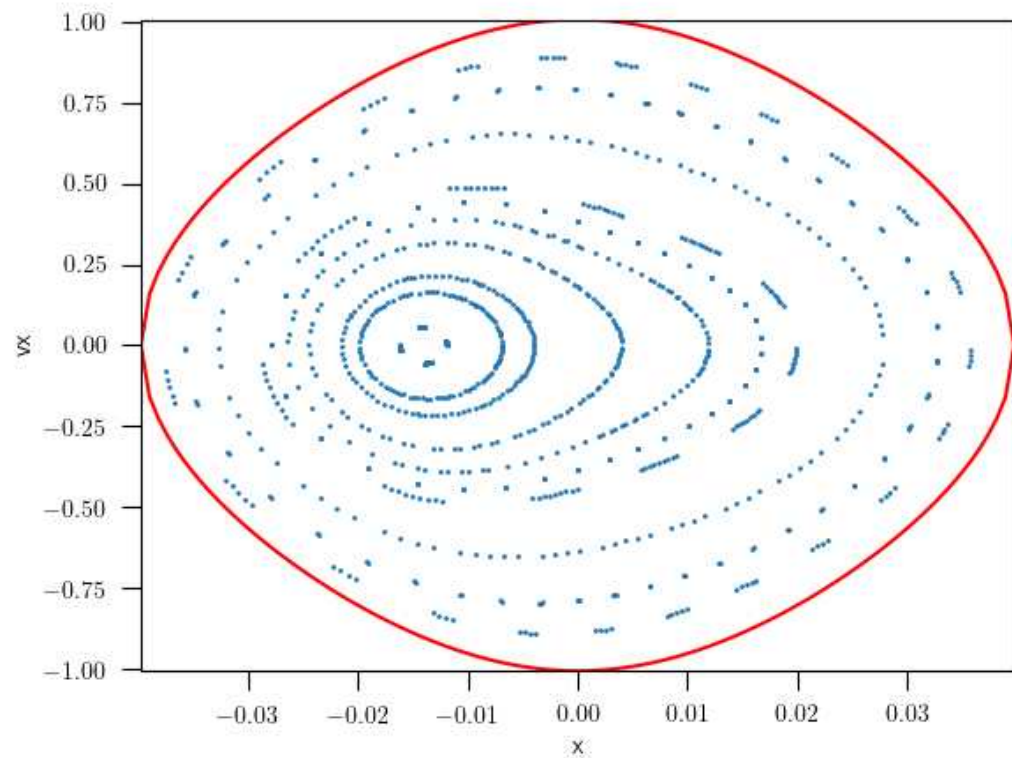
$$\Omega = 0$$



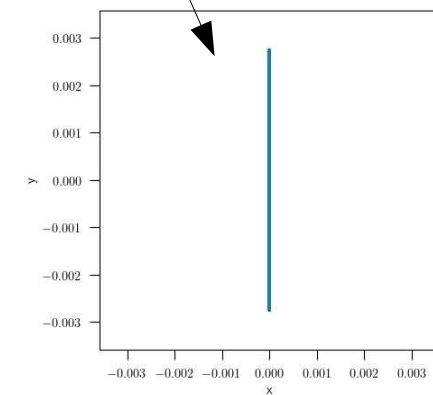
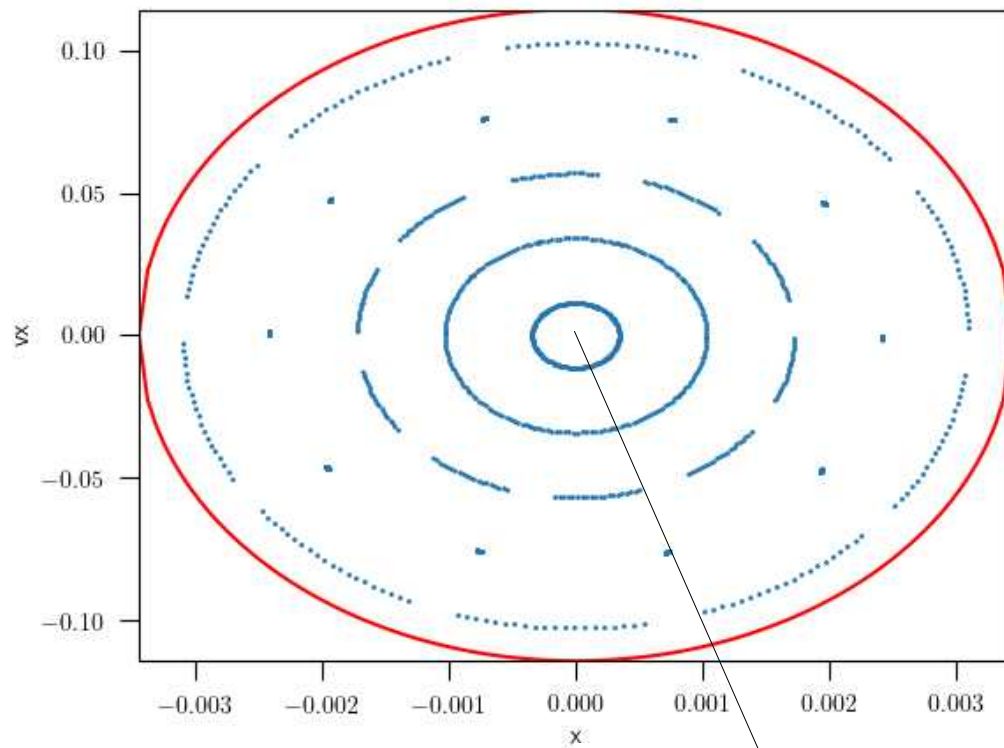
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Orbits around L_3

$$\Omega = 1$$



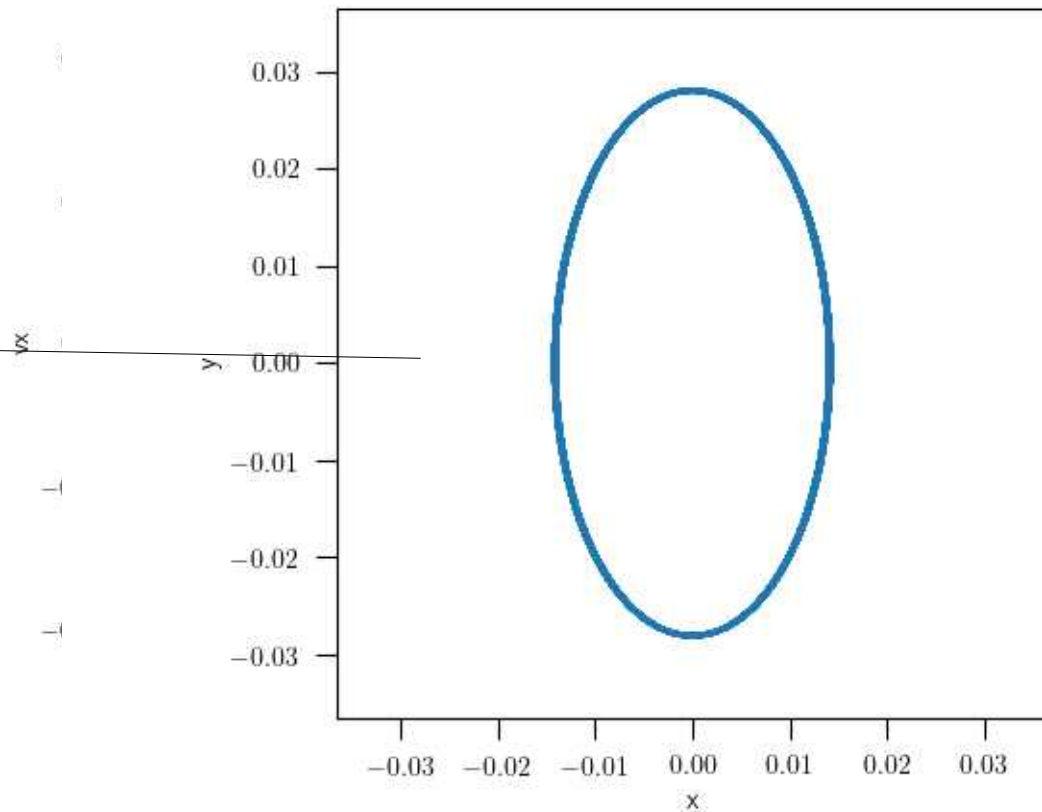
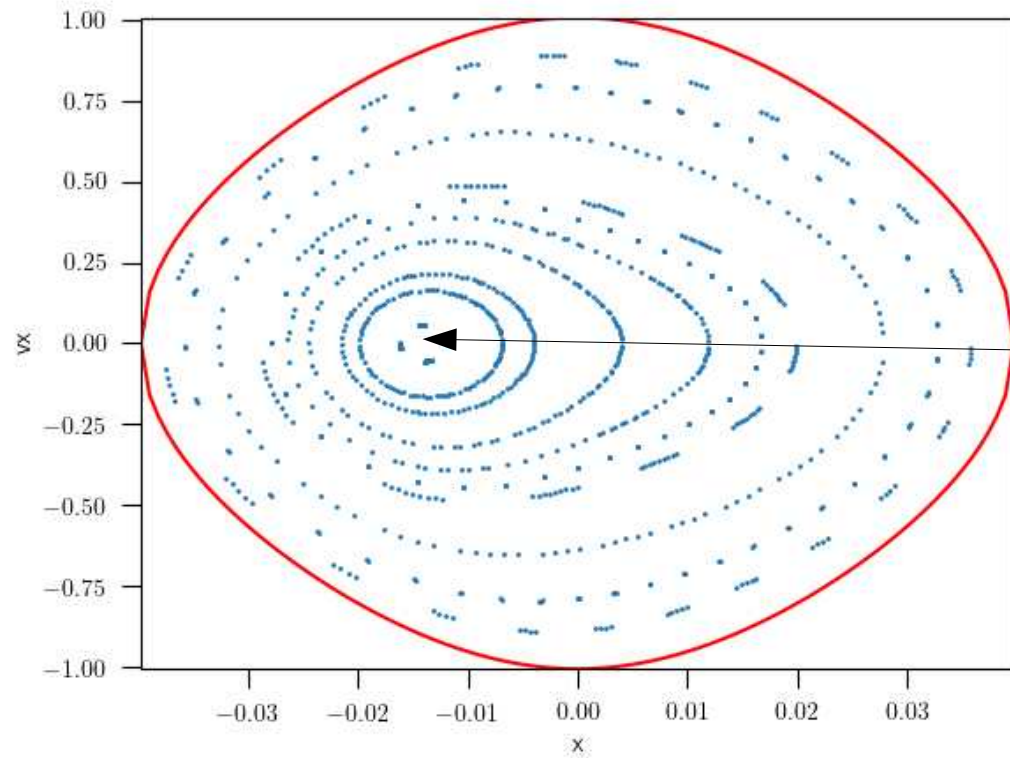
$$\Omega = 0$$



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Short axis (Y) orbits (periodic)

$$\Omega = 1$$



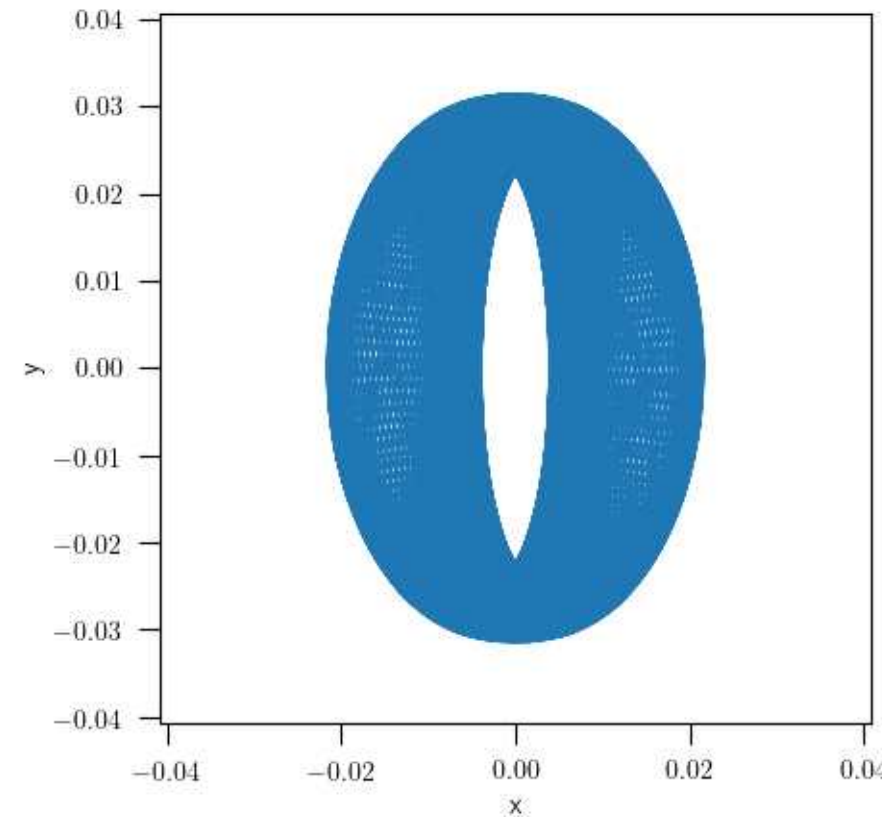
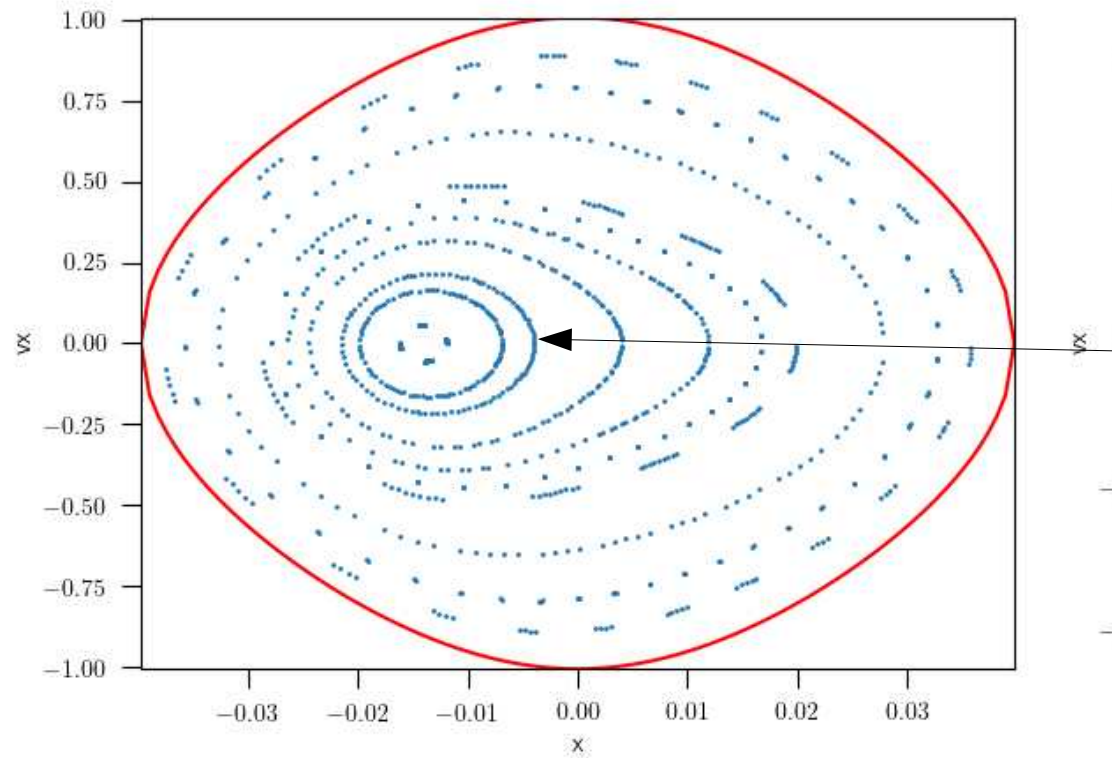
X4

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3 --x -0.014
```

Apparition of a periodic loop orbit
(replace the radial orbit, perpendicular to the bar), clockwise rotation

Short axis (Y) orbits (periodic)

$$\Omega = 1$$

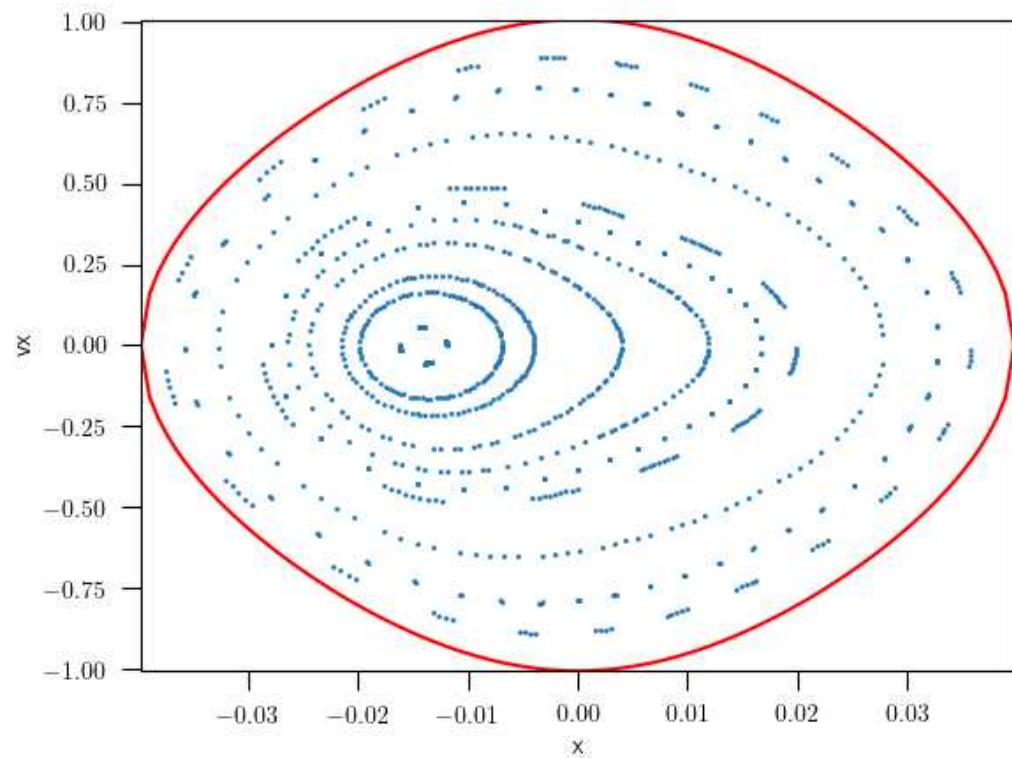


X4

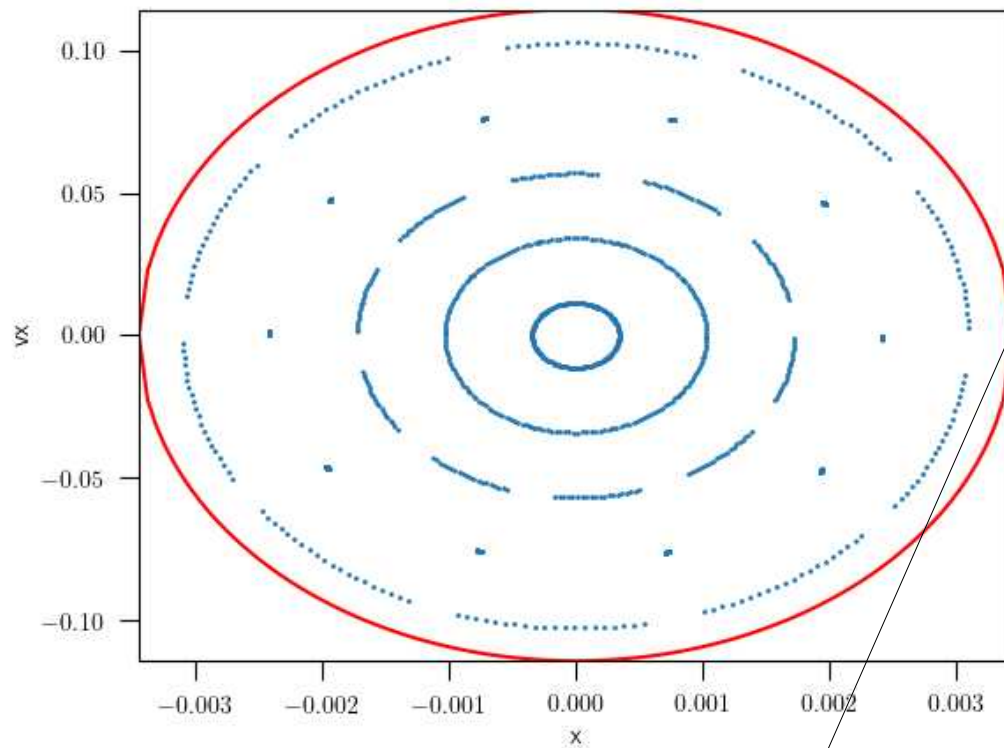
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3 --x -0.004
```

Orbits around L_3

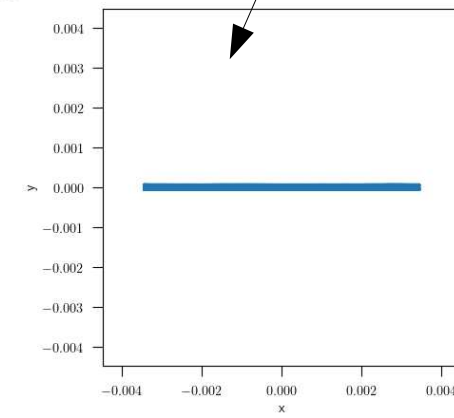
$$\Omega = 1$$



$$\Omega = 0$$

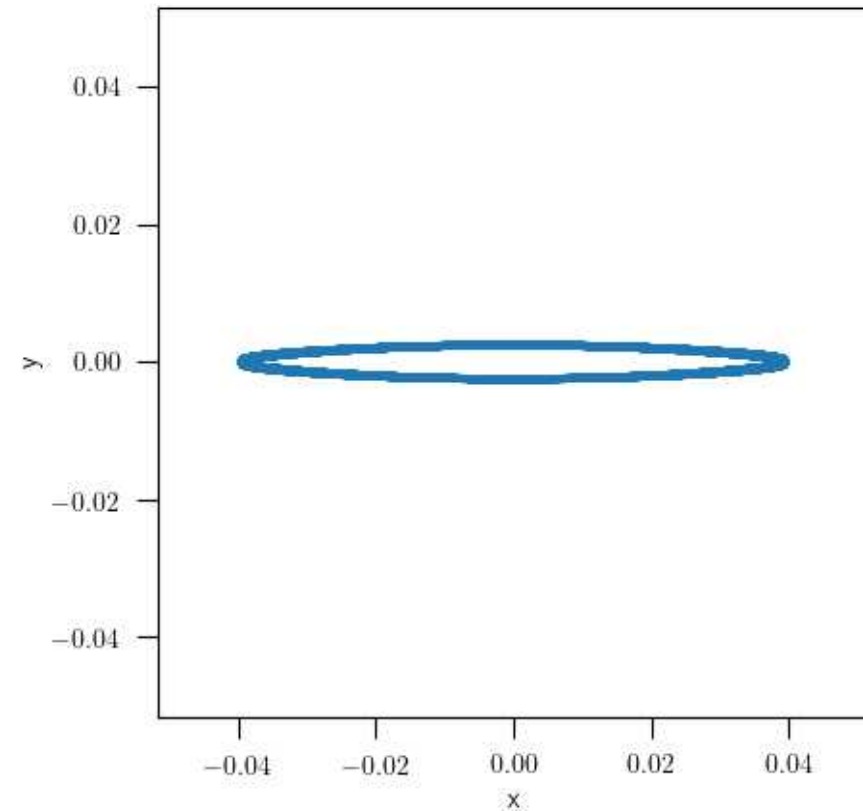
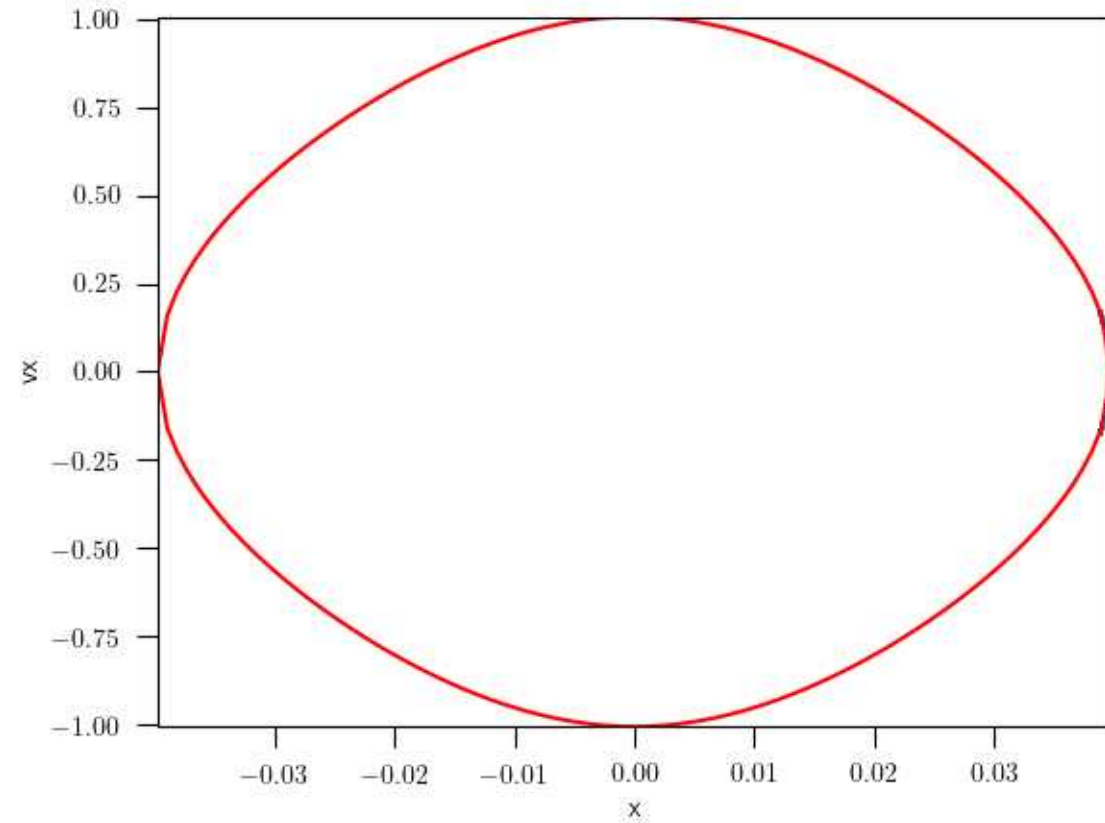


```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```



Long axis (X) orbits (periodic)

$$\Omega = 1$$



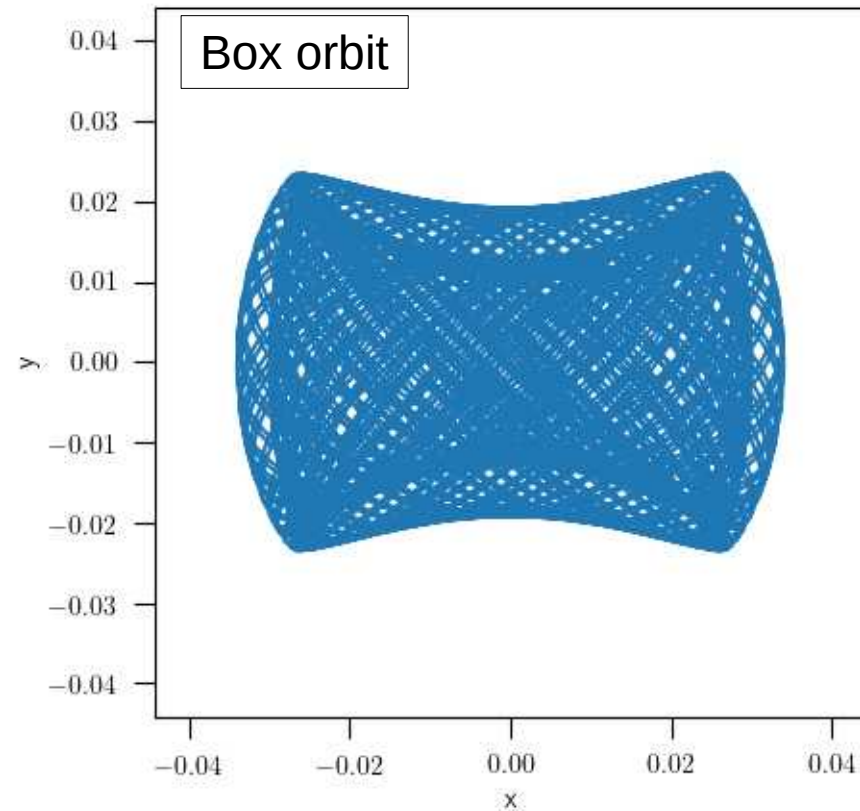
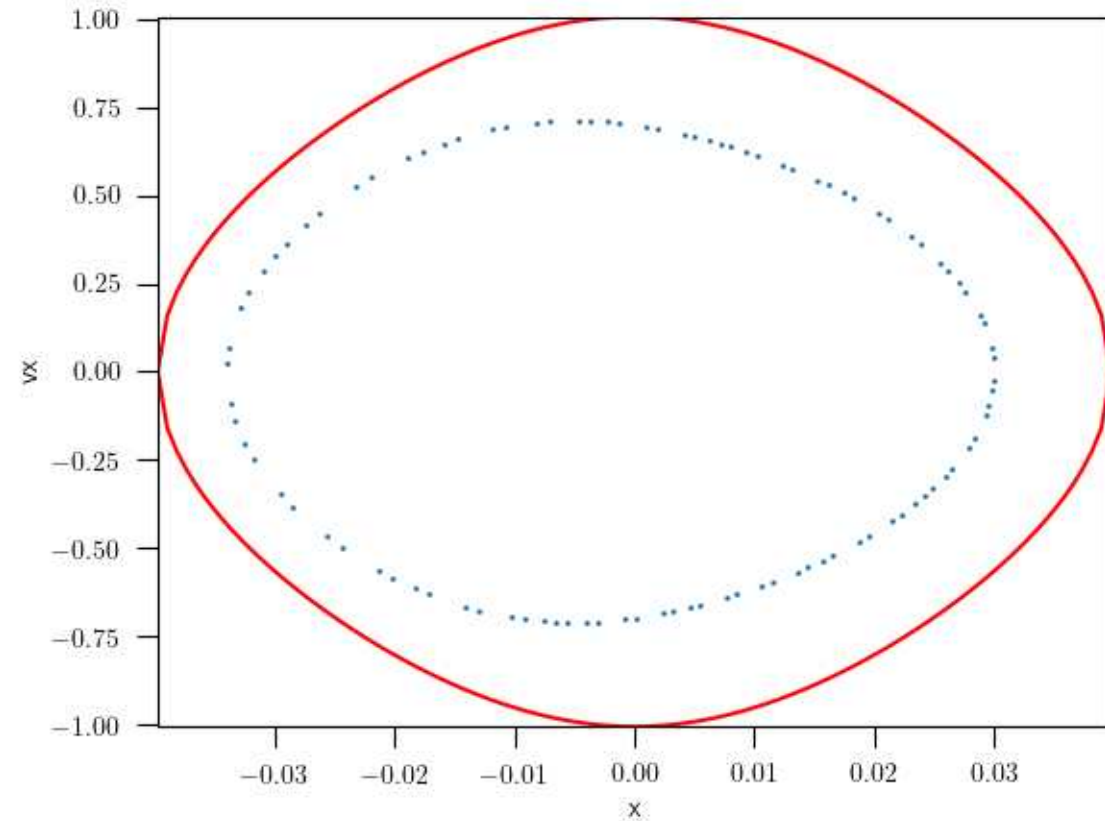
x1

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3 --x 0.03975
```

Apparition of a periodic loop orbit
(replace the radial orbit, parallel
to the bar), anti-clockwise rotation

Long axis (X) orbits (non periodic)

$$\Omega = 1$$

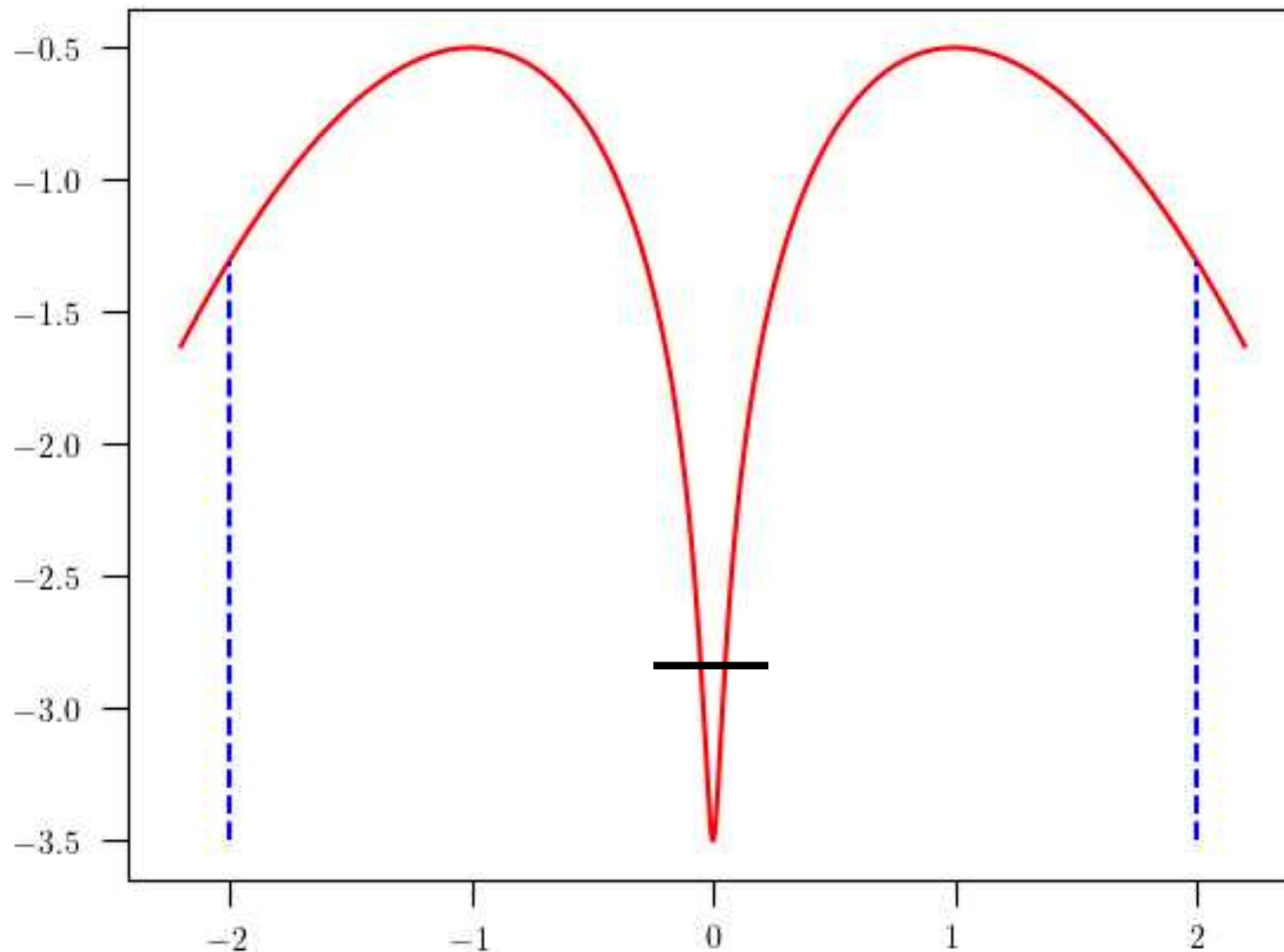


x1

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3 --x 0.03
```

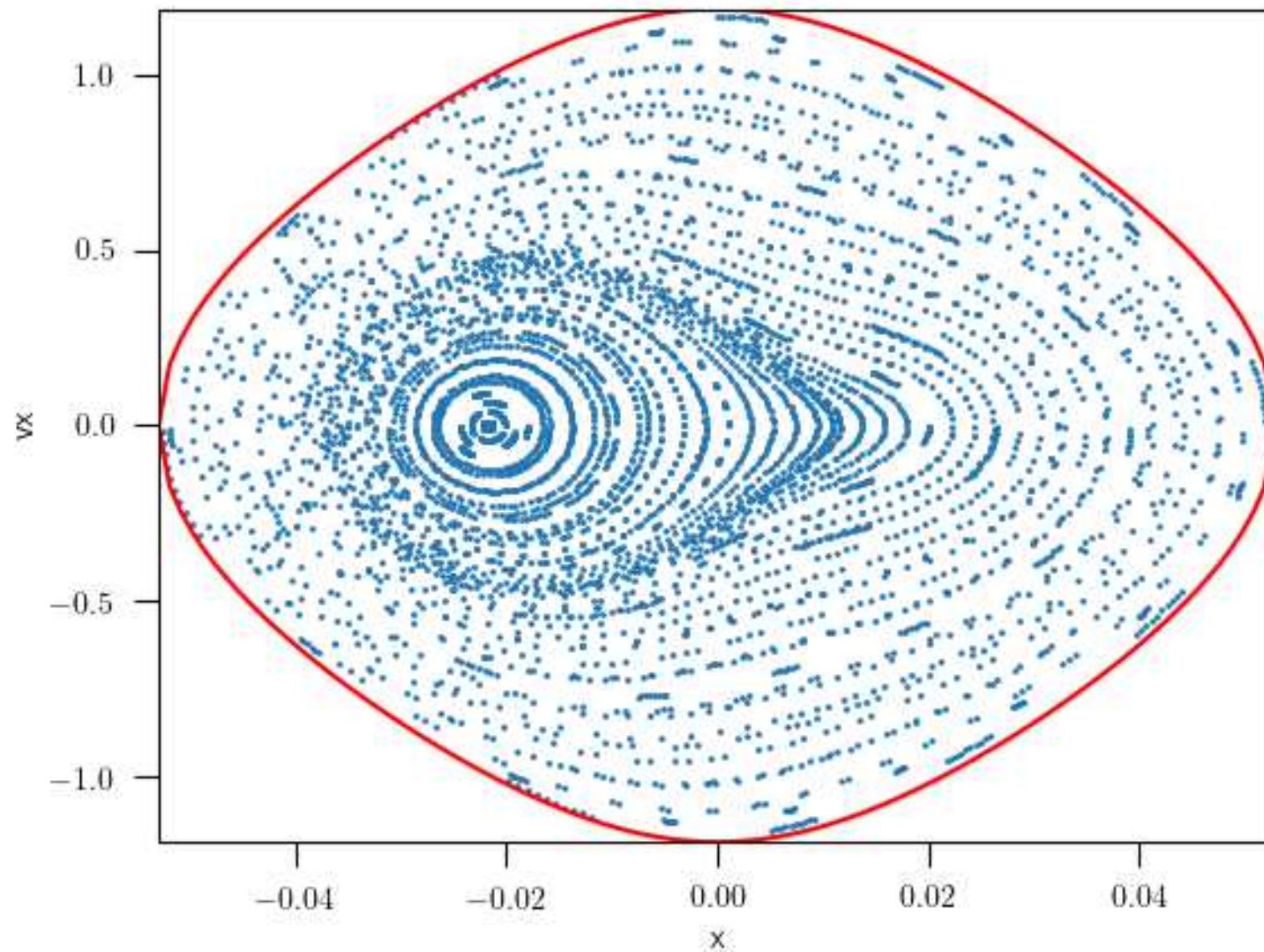
Increasing the energy

$$E = -2.8$$



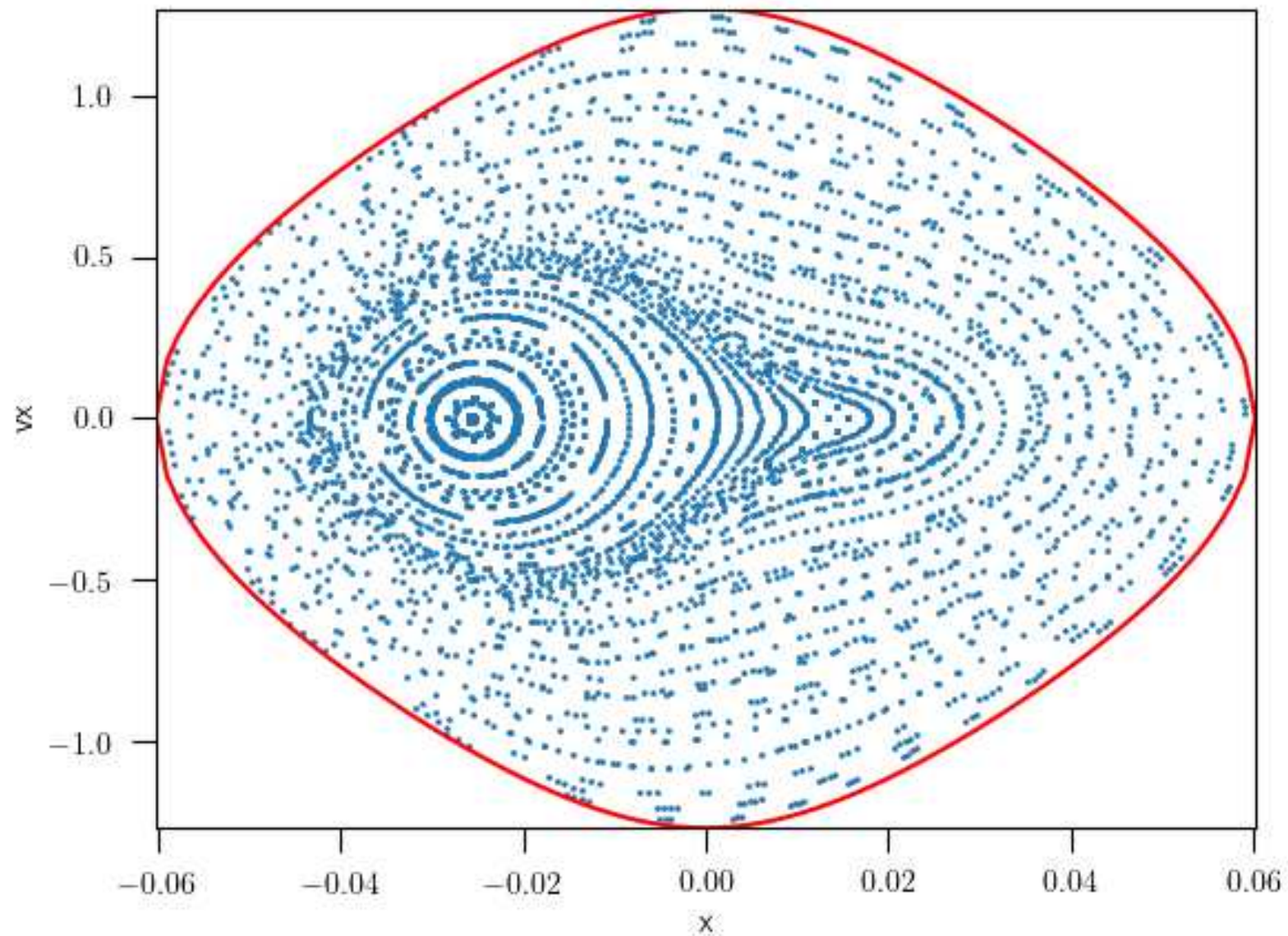
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

$$E = -2.8$$



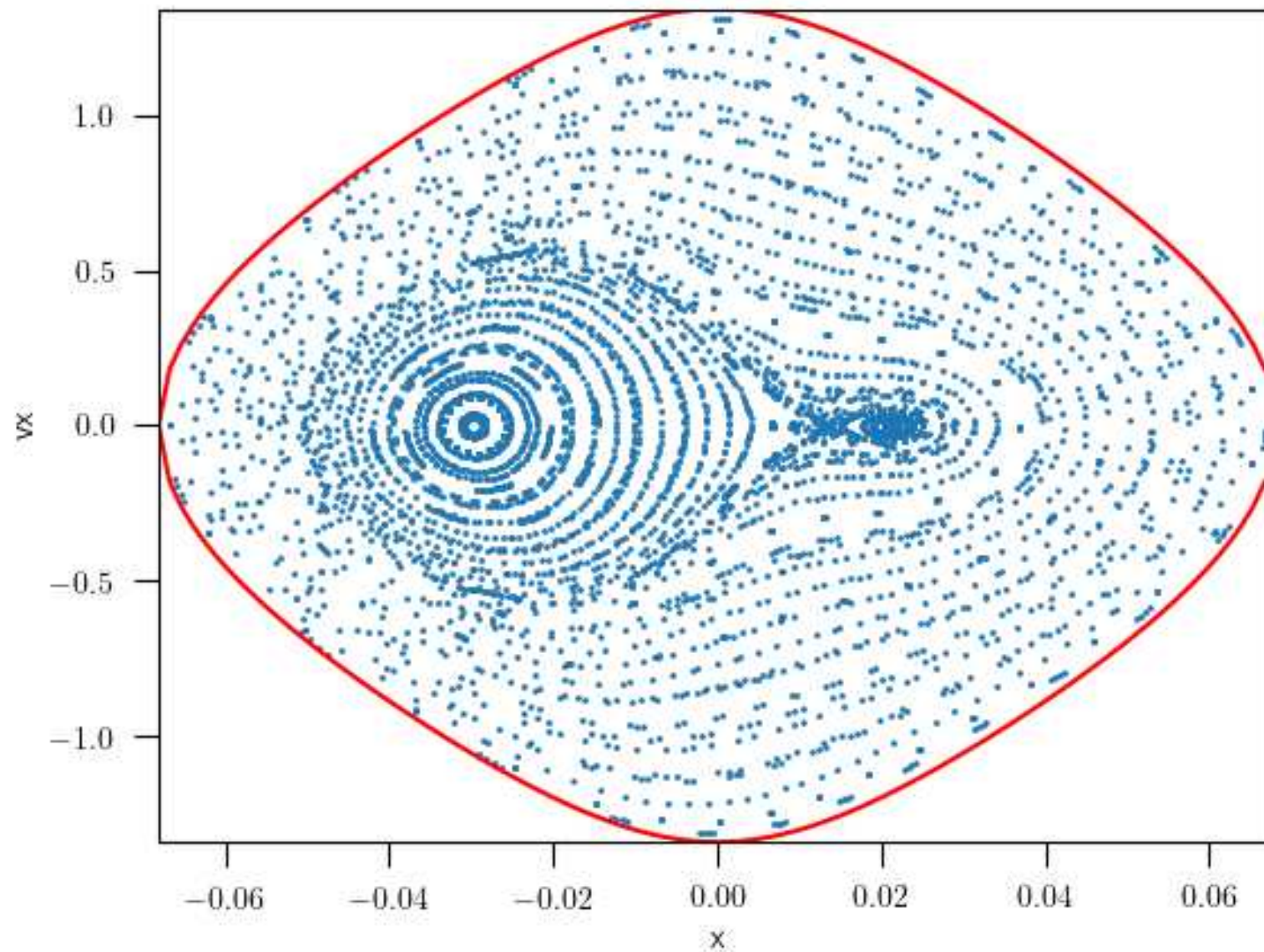
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.8 --norbits 50
```

$$E = -2.7$$



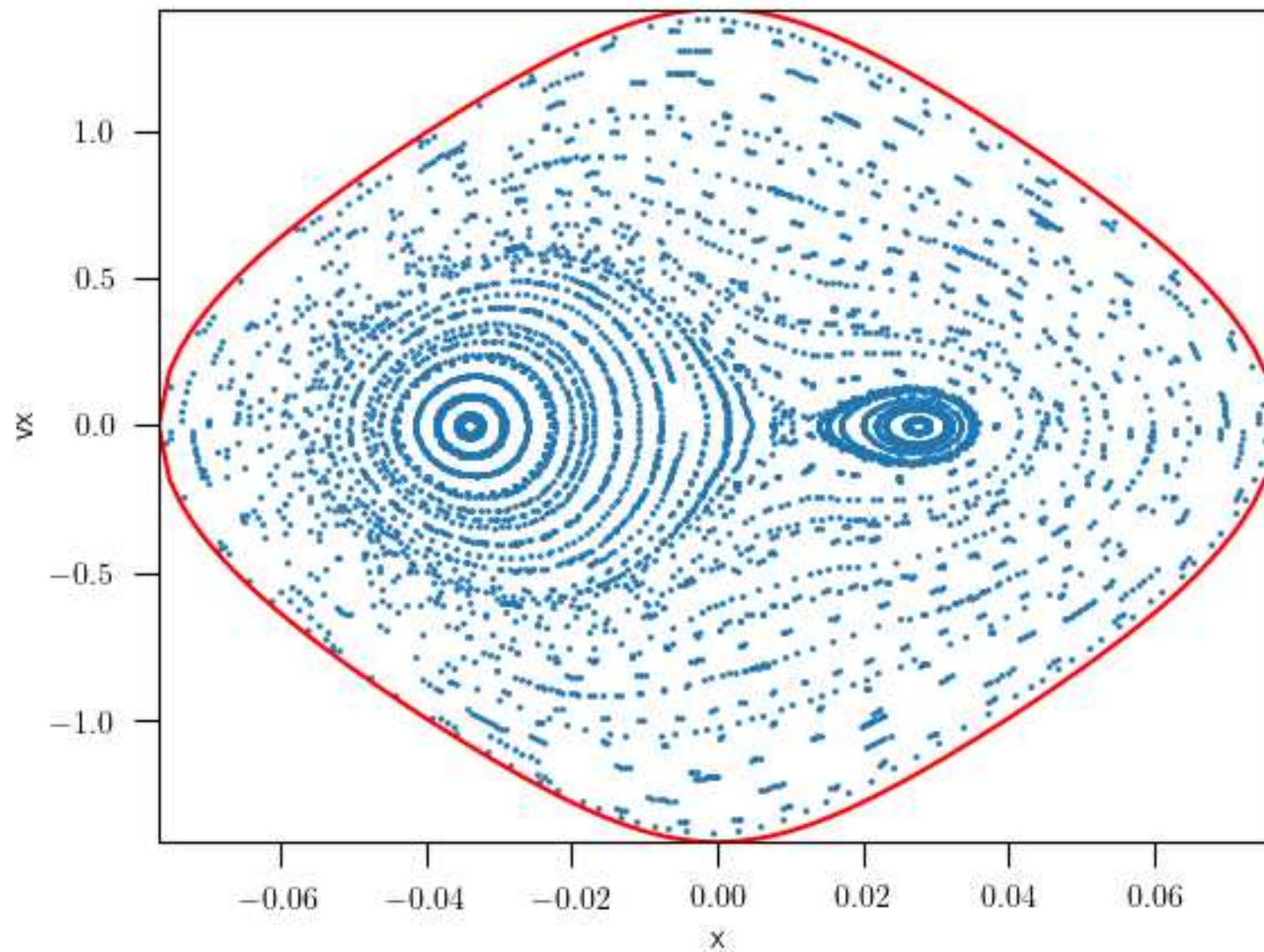
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.7 --norbits 50
```

$$E = -2.6$$



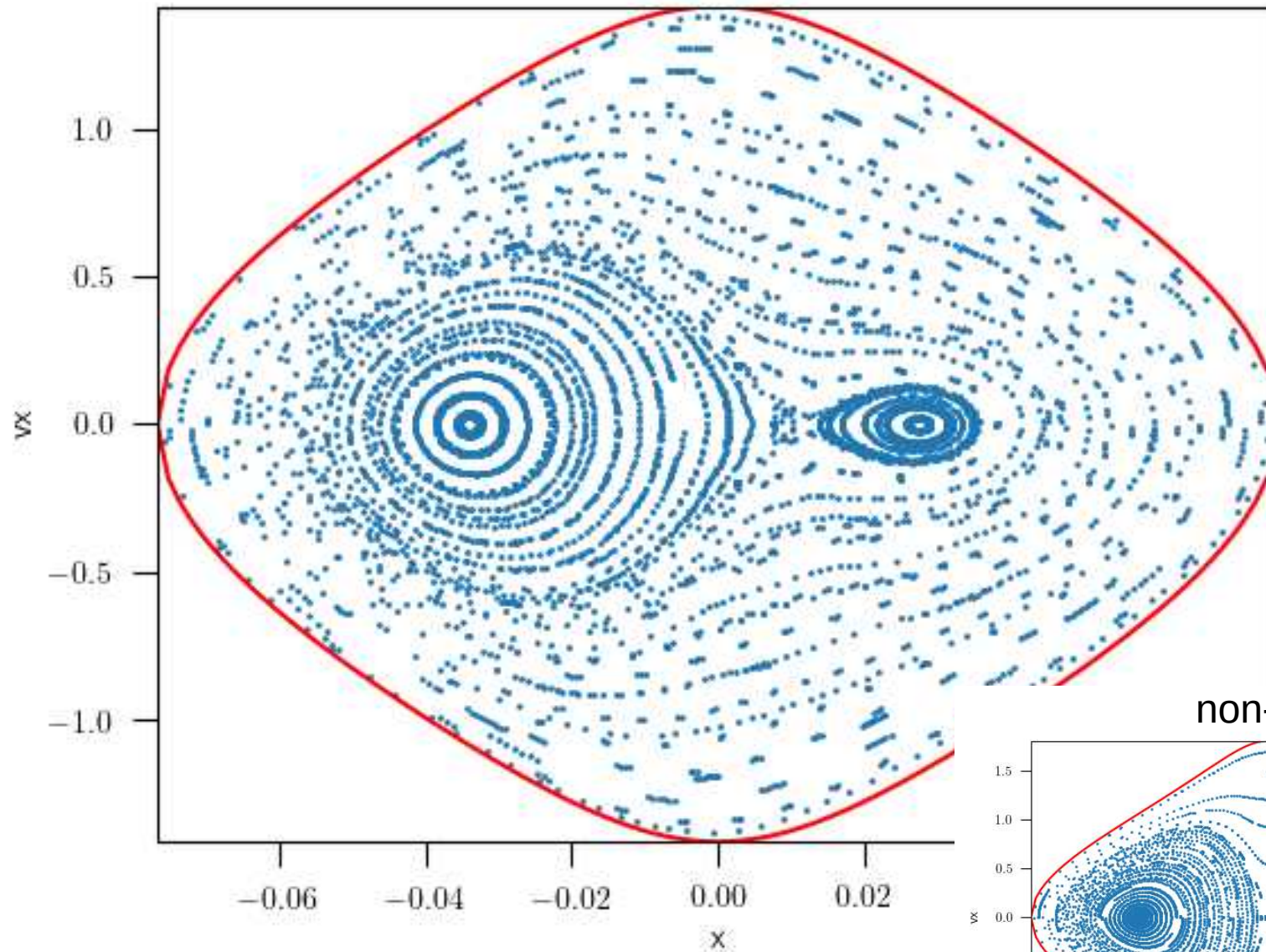
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.6 --norbits 50
```

$$E = -2.5$$

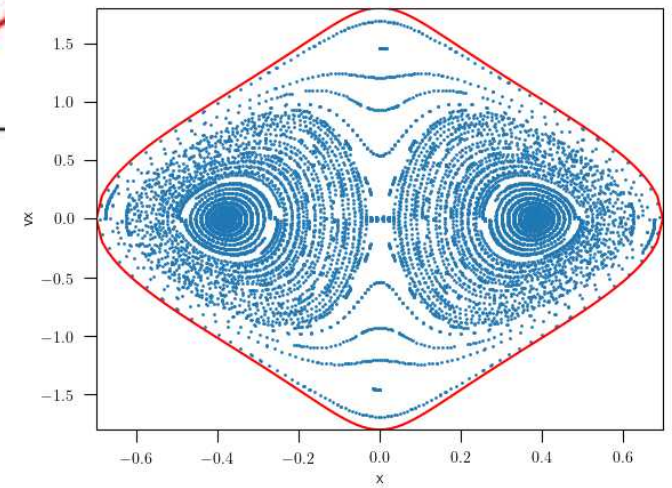


```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

$$E = -2.5$$

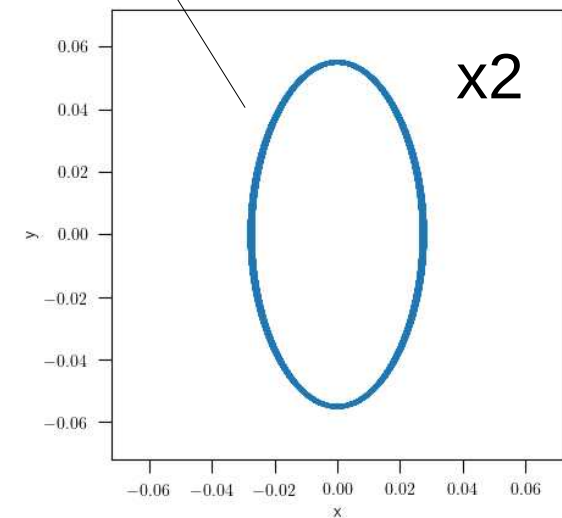
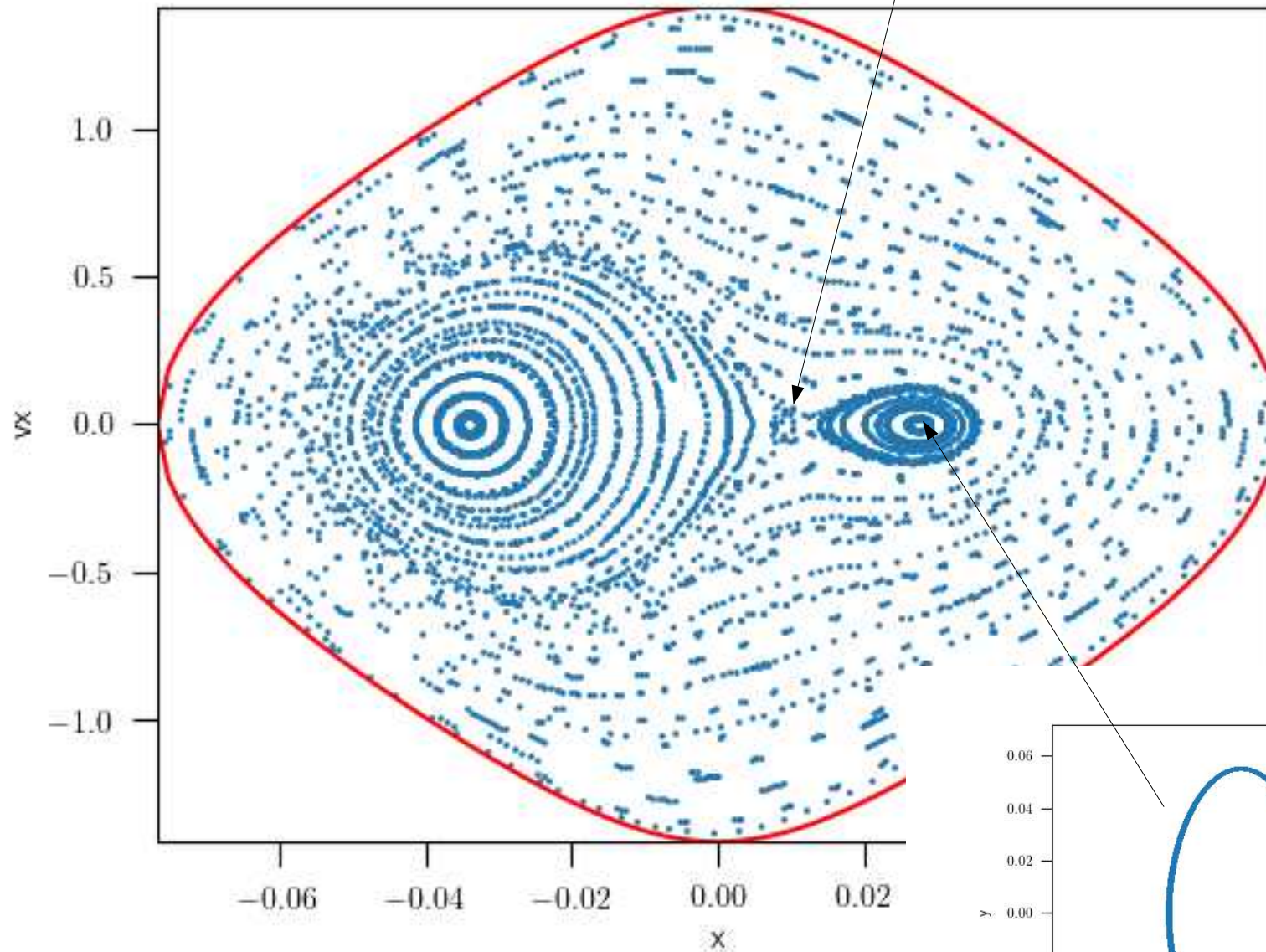


non-rotating case



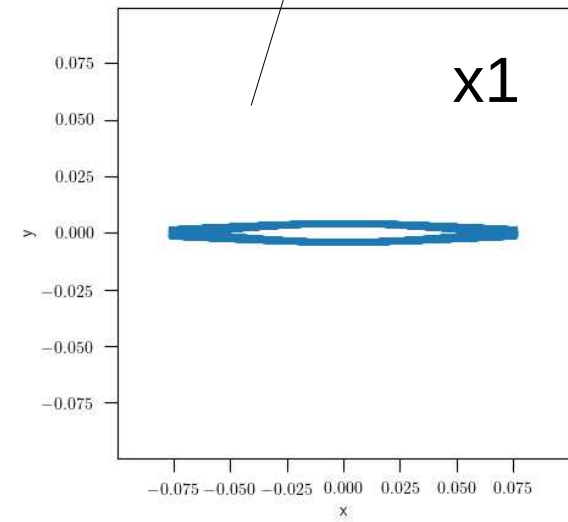
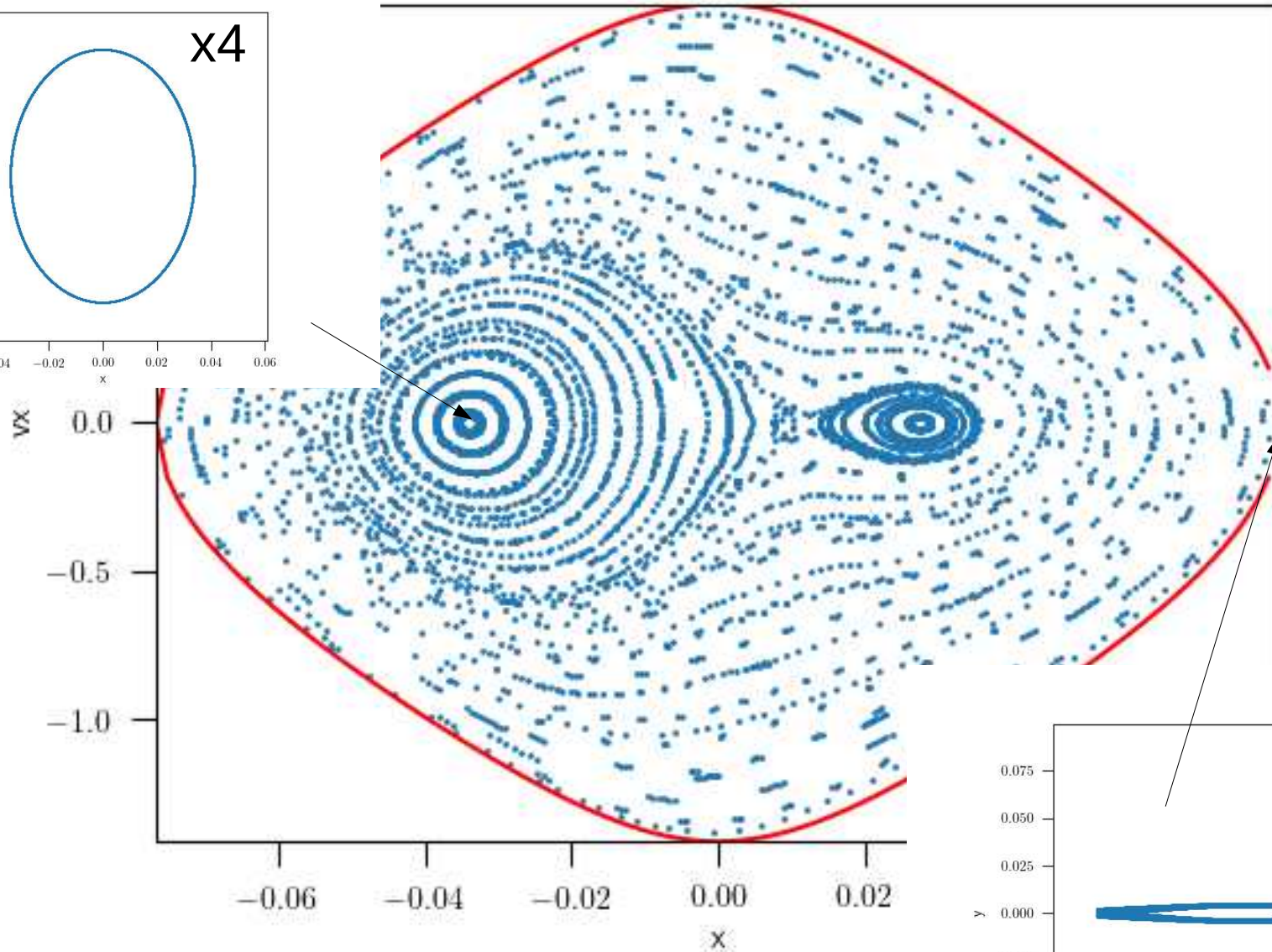
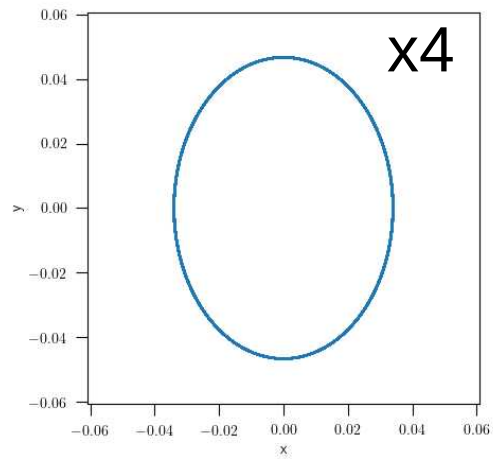
`./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50`

Bifurcation : apparition of x_2 (stable)/ x_3 (unstable) orbits



`./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0268`

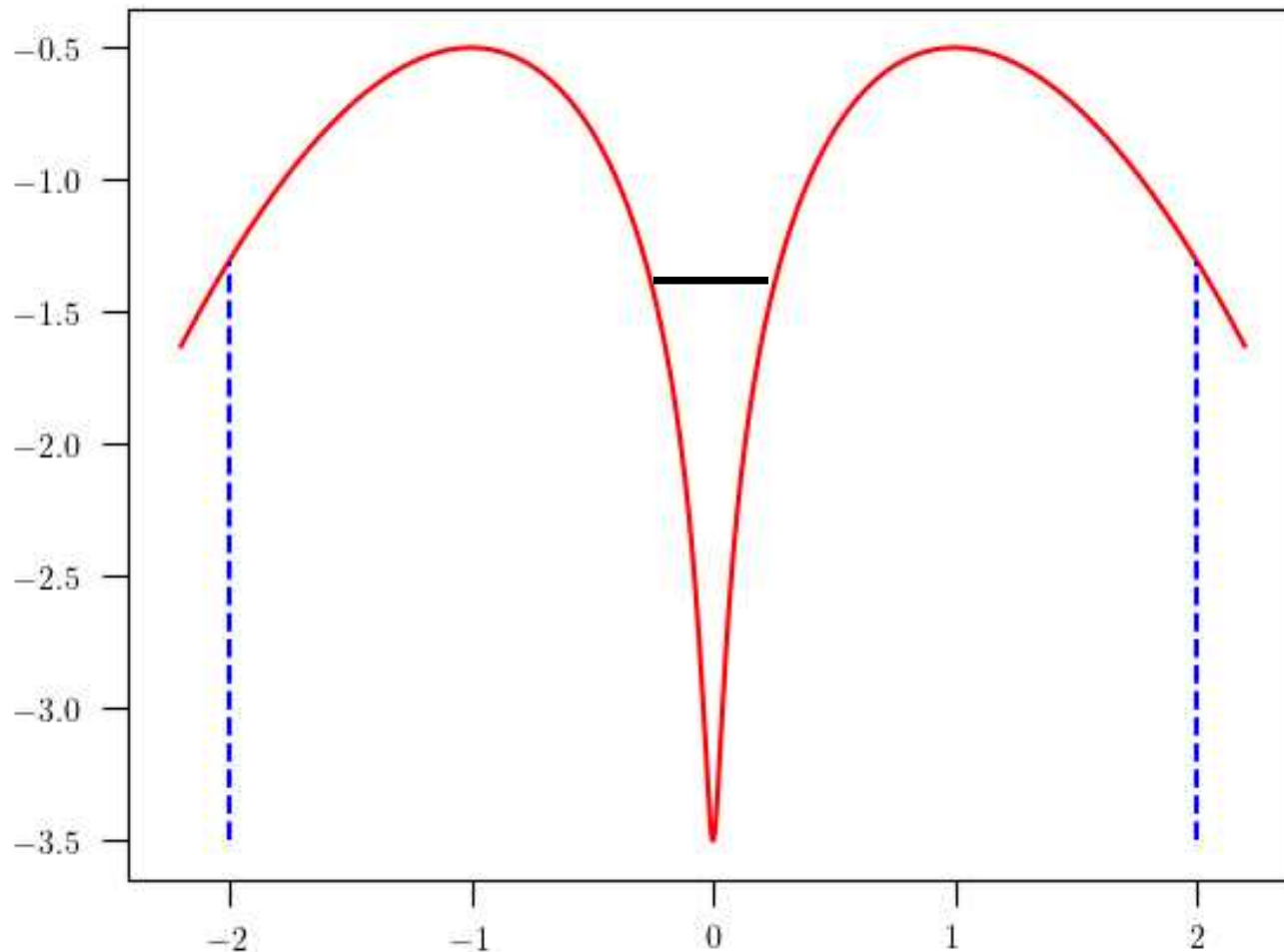
x1 : prograde x4 : retrograde



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0766659  
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x -0.034
```

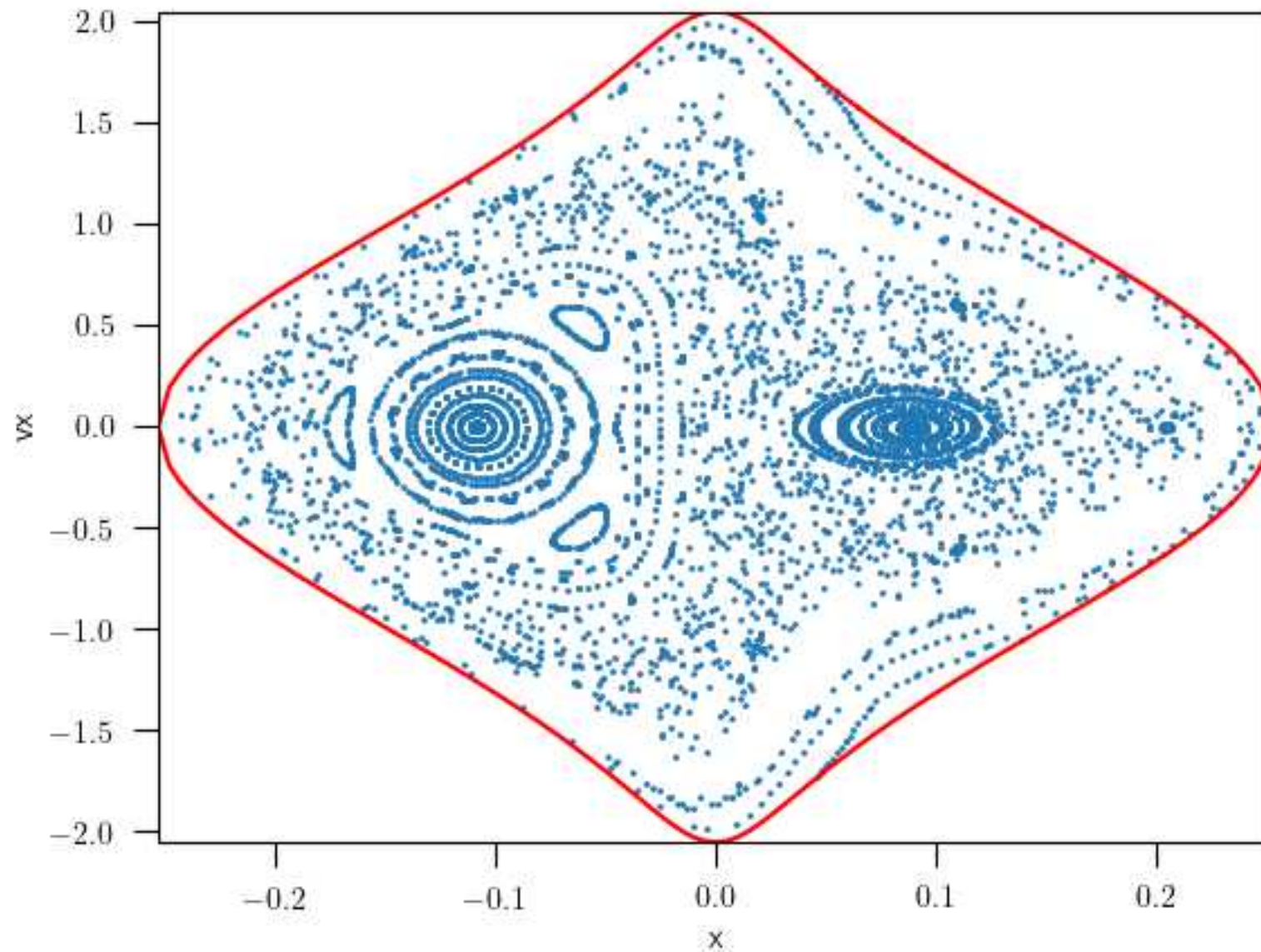
Increasing the energy

$$E = -1.4$$



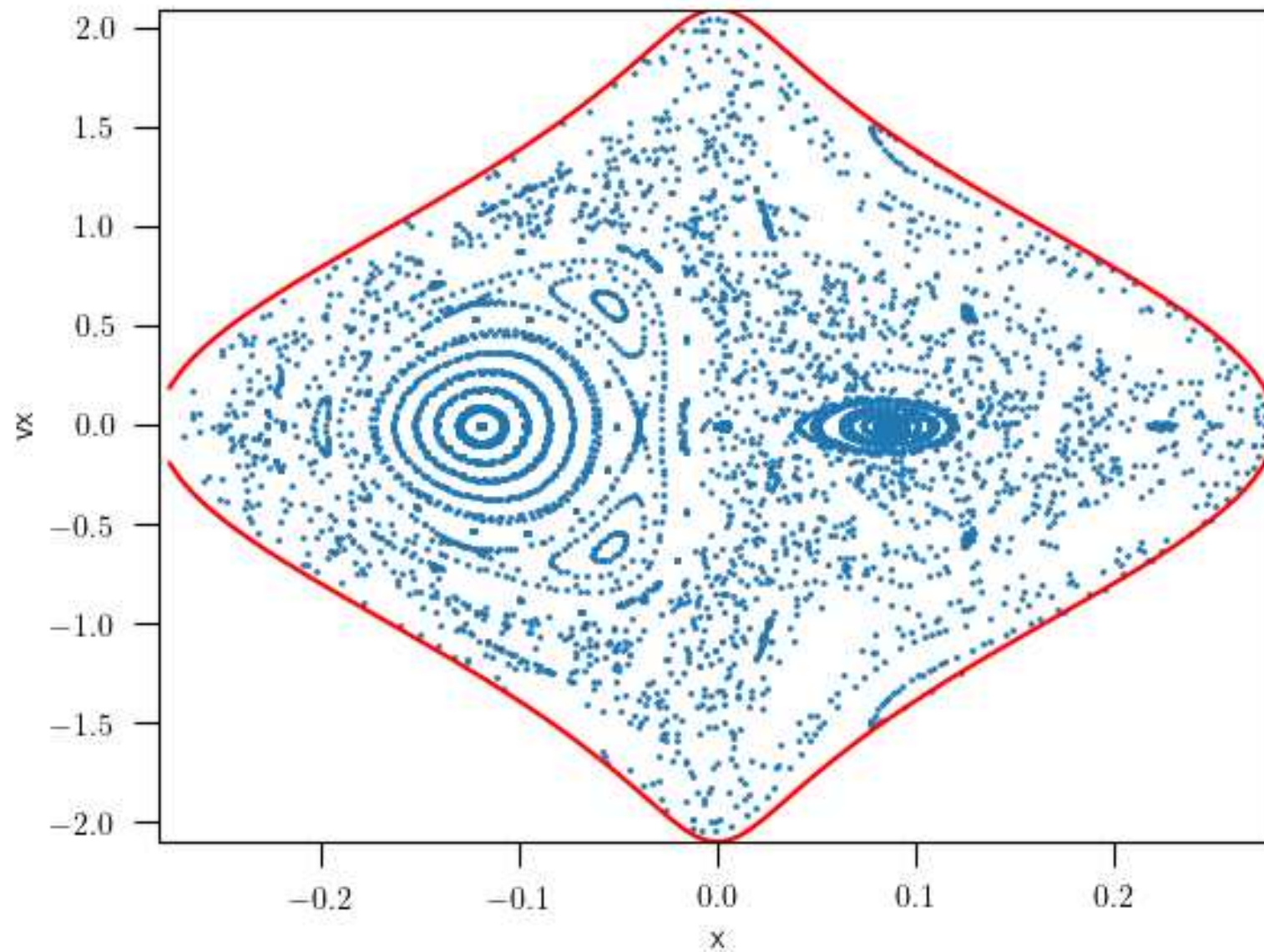
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

$$E = -1.4$$



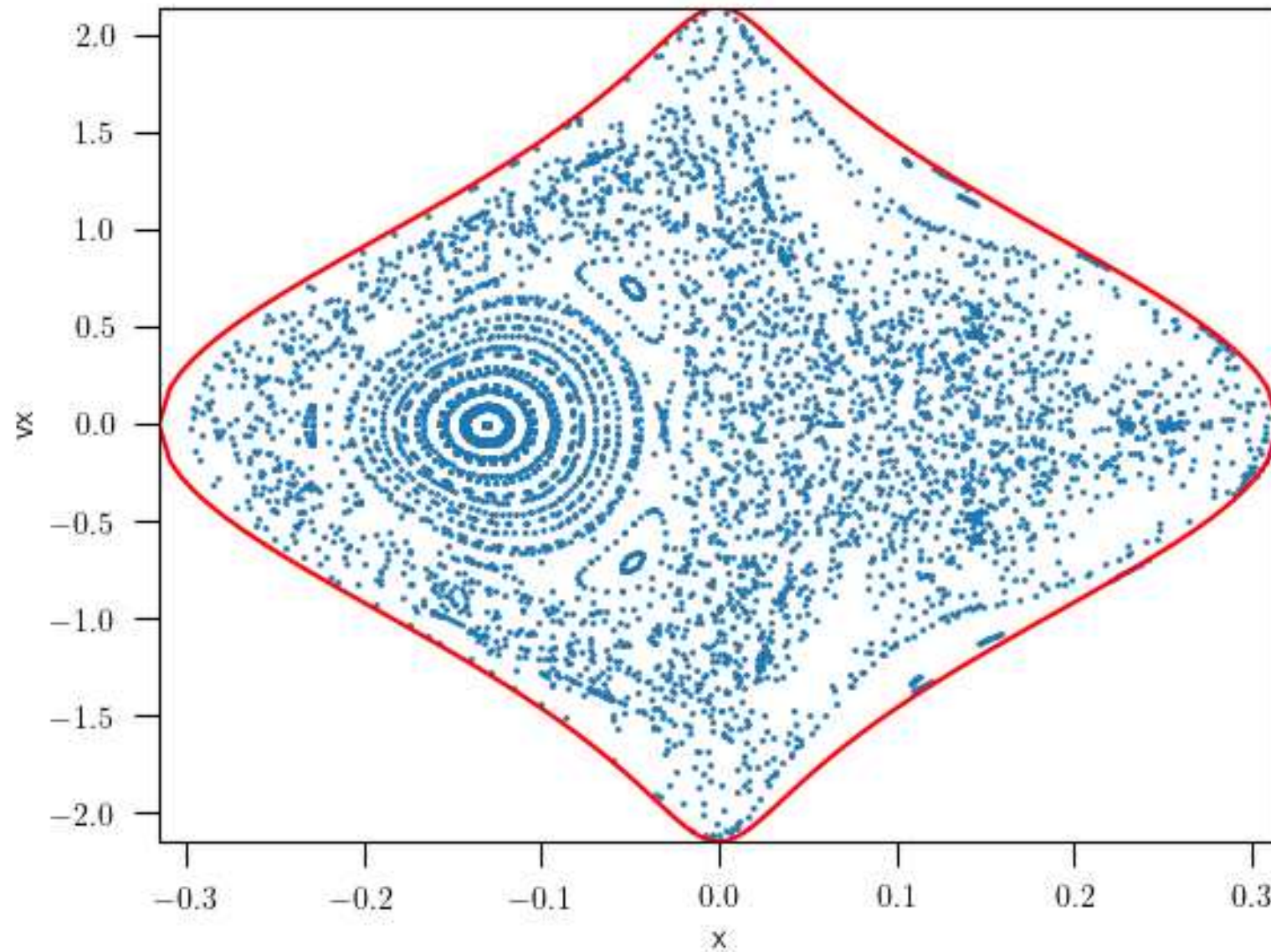
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.4 --norbits 50
```

$$E = -1.3$$



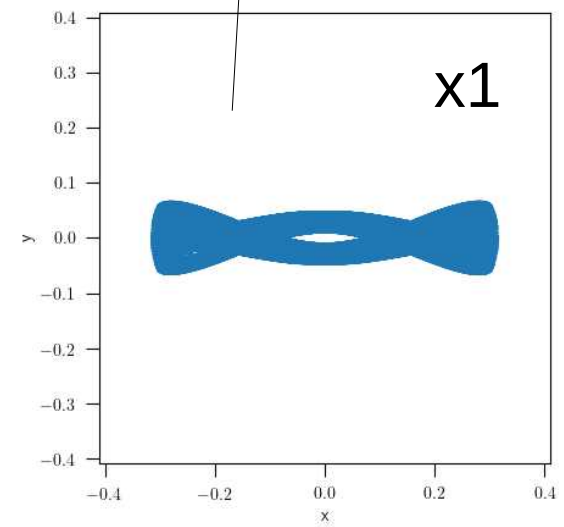
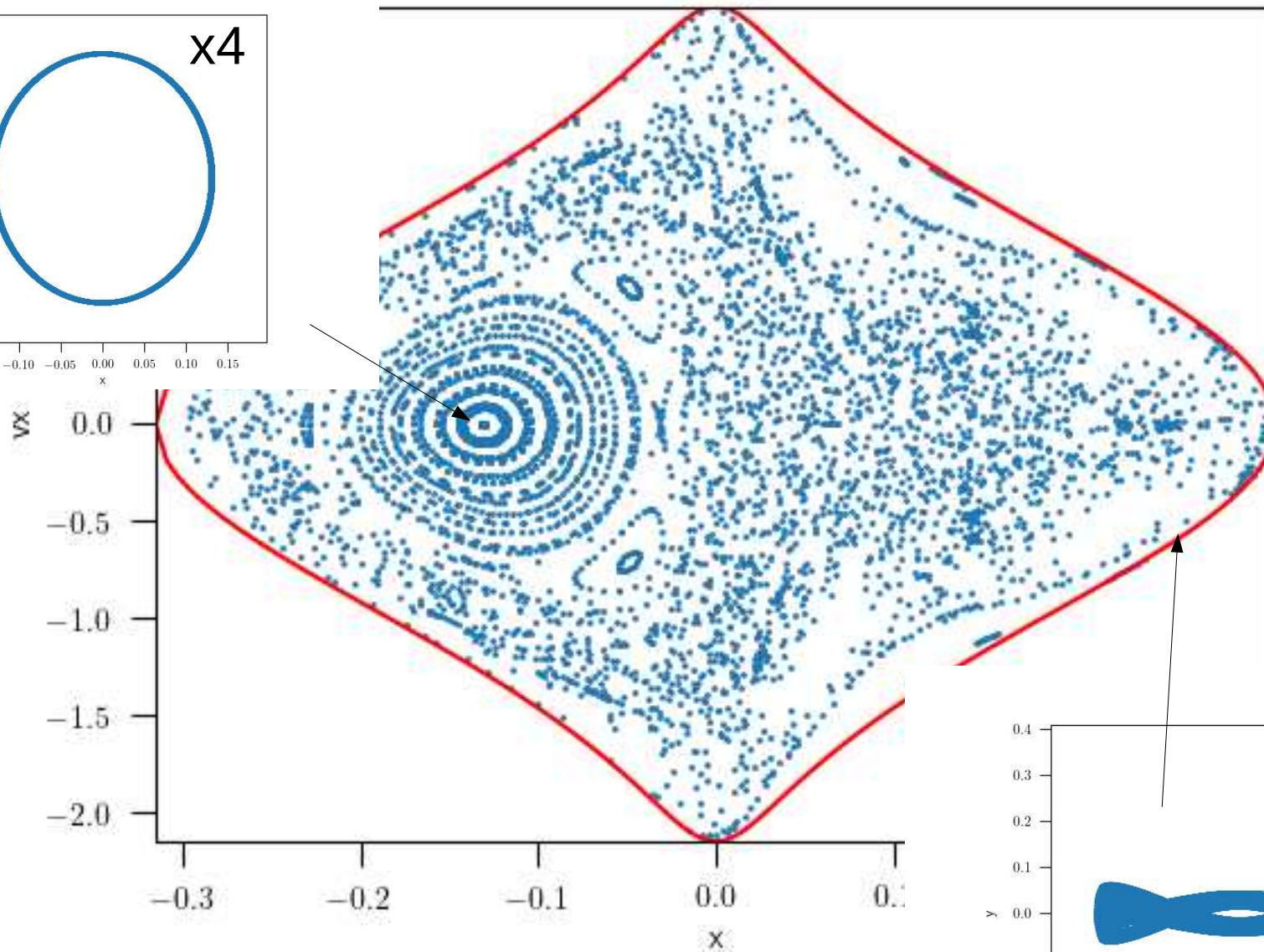
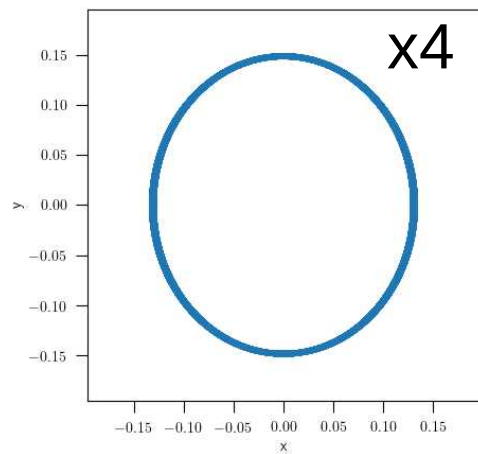
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.3 --norbits 50
```

$E = -1.2$
Bifurcation : x_2/x_3 disappeared



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --norbits 50
```

$$E = -1.2$$

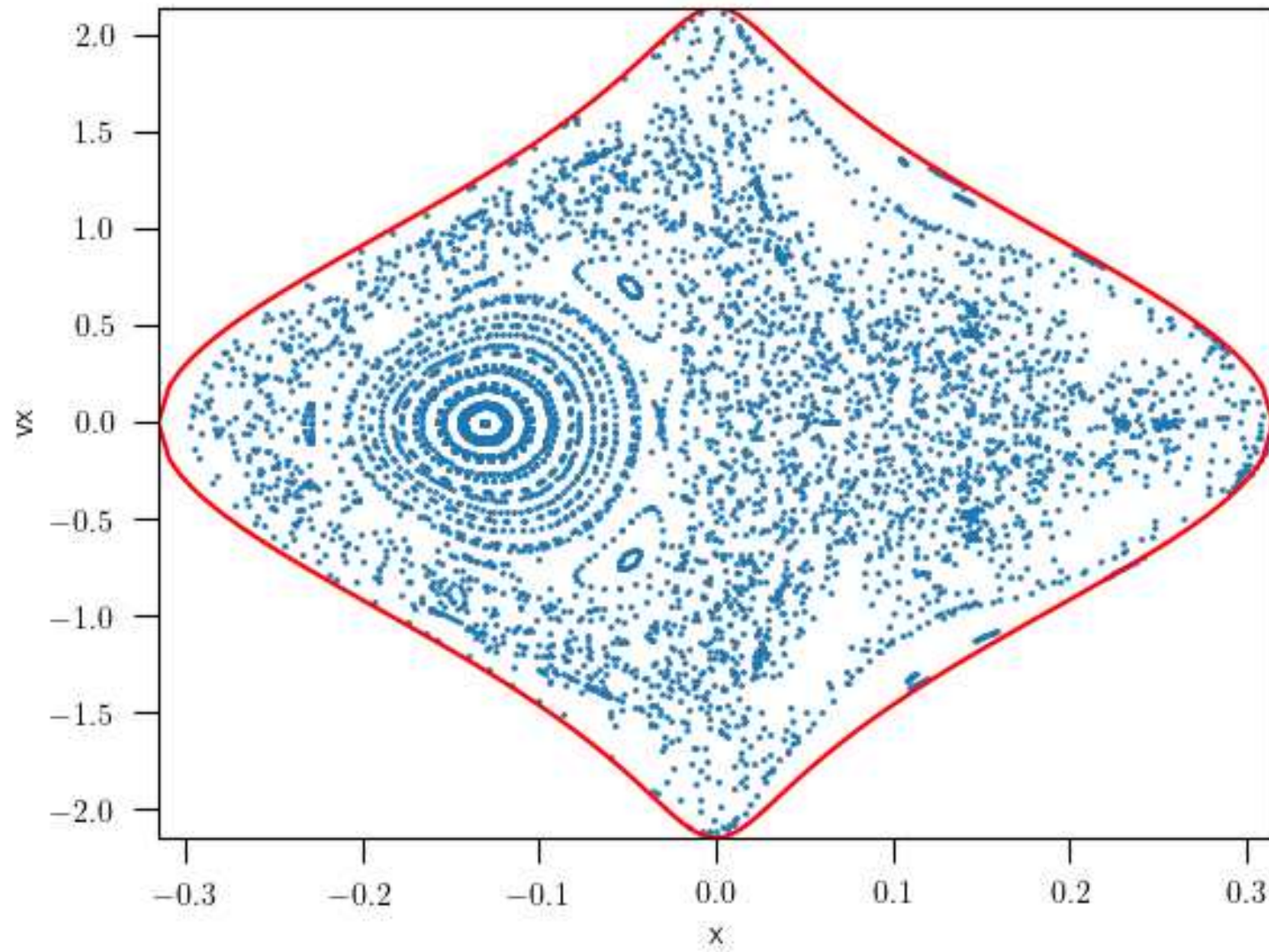


```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
```

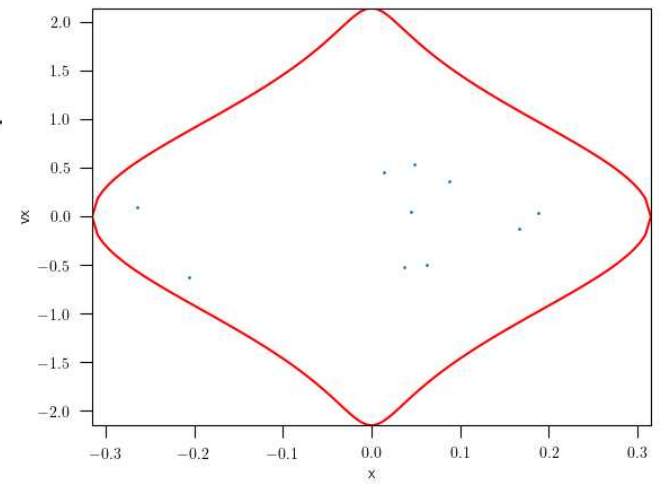
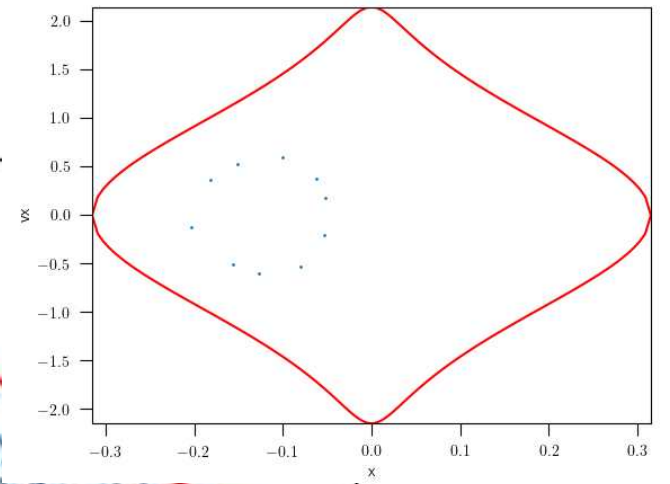
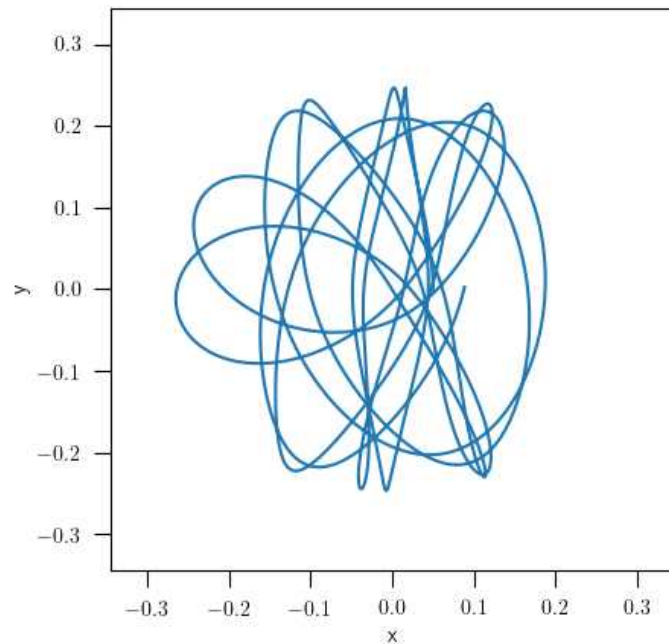
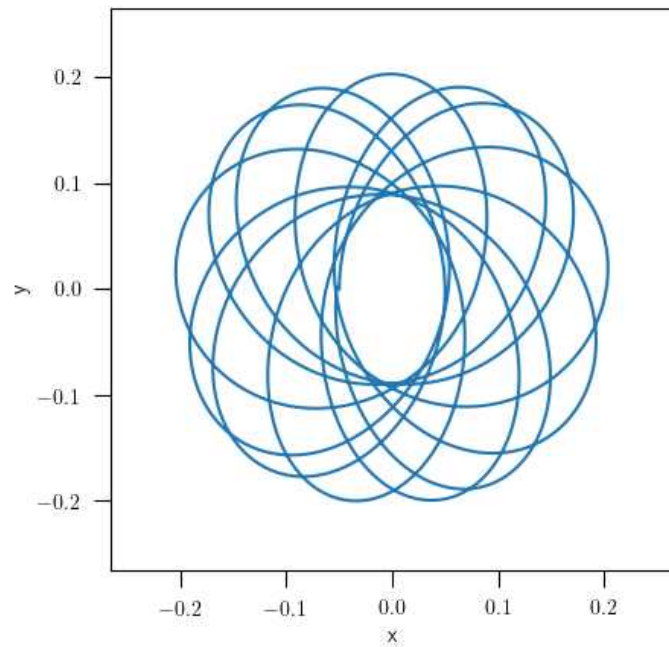
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x -0.1283
```

Chaotic orbits

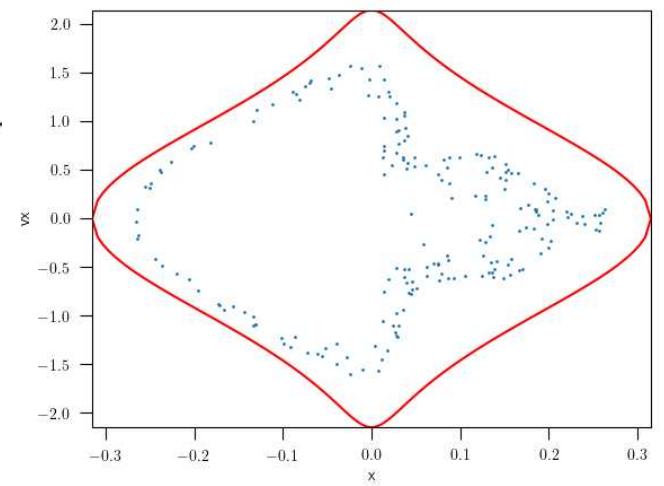
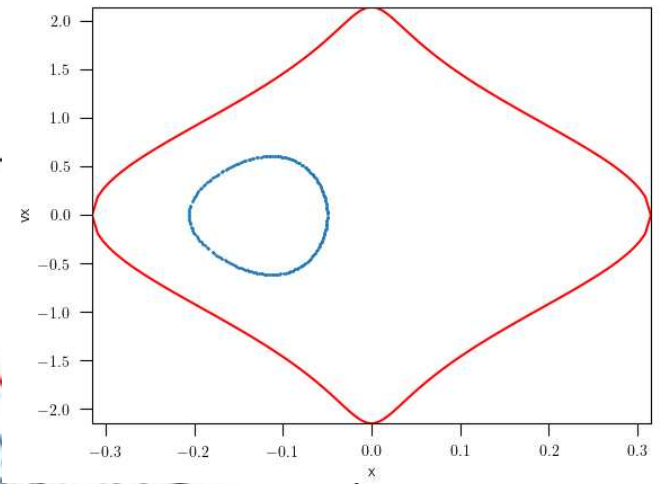
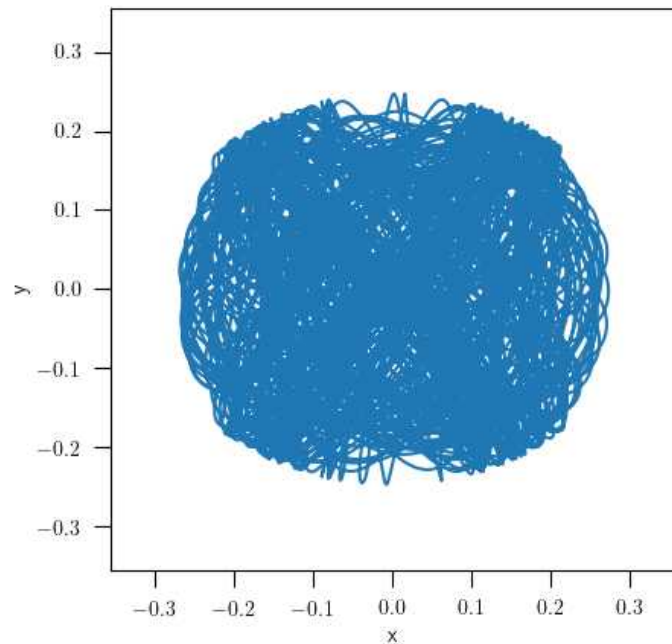
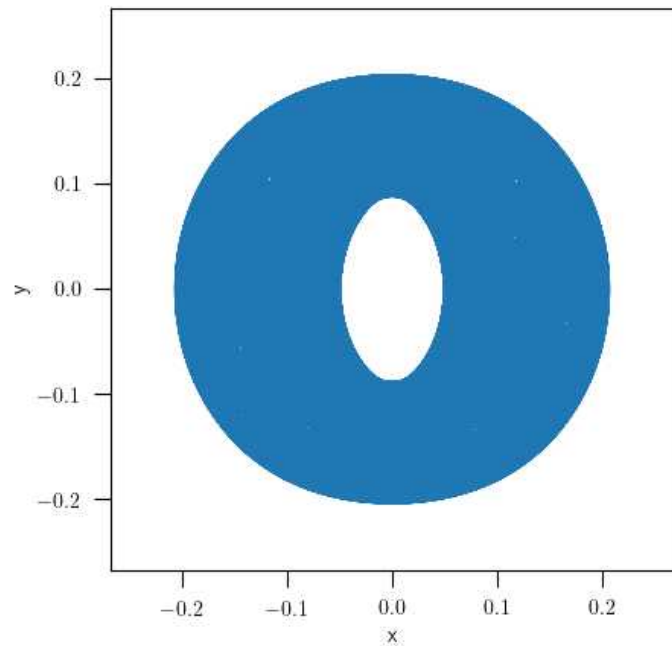
Chaotic orbits



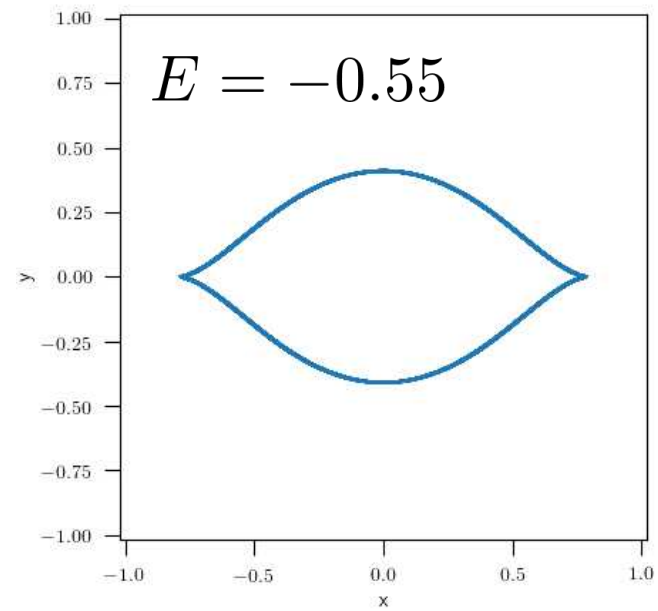
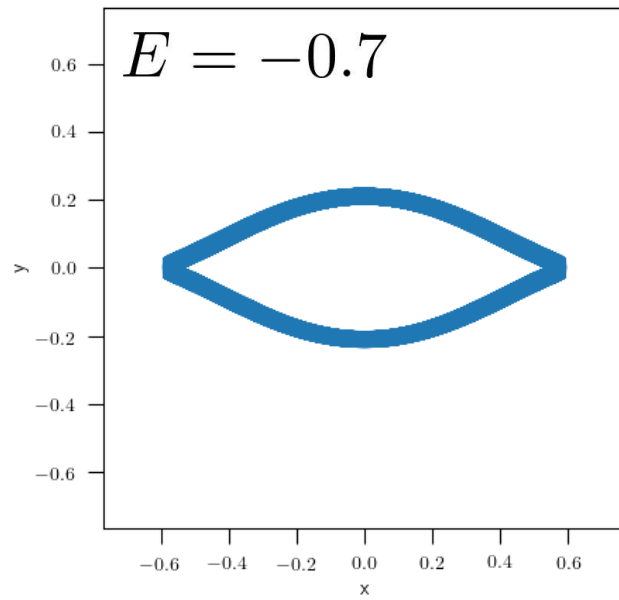
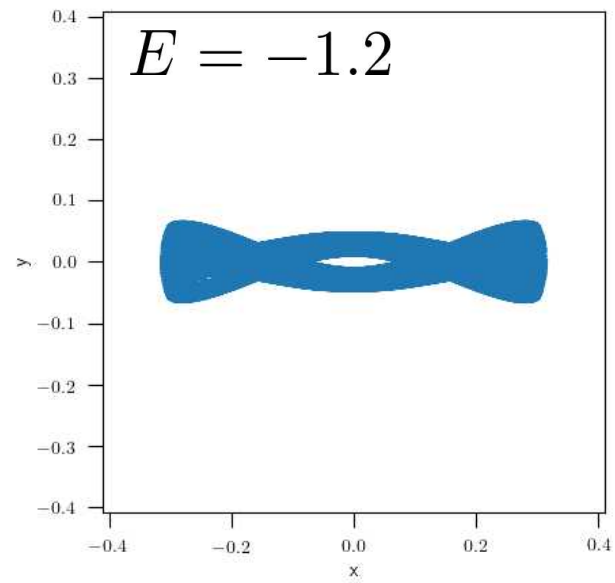
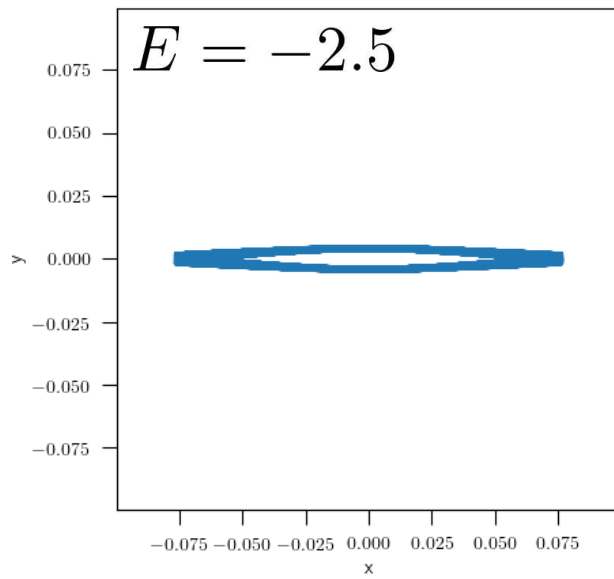
chaotic orbits



chaotic orbits



Evolution of the x_1 orbit with increasing energy



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.7 --x 0.590356
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.55 --x 0.783882
```

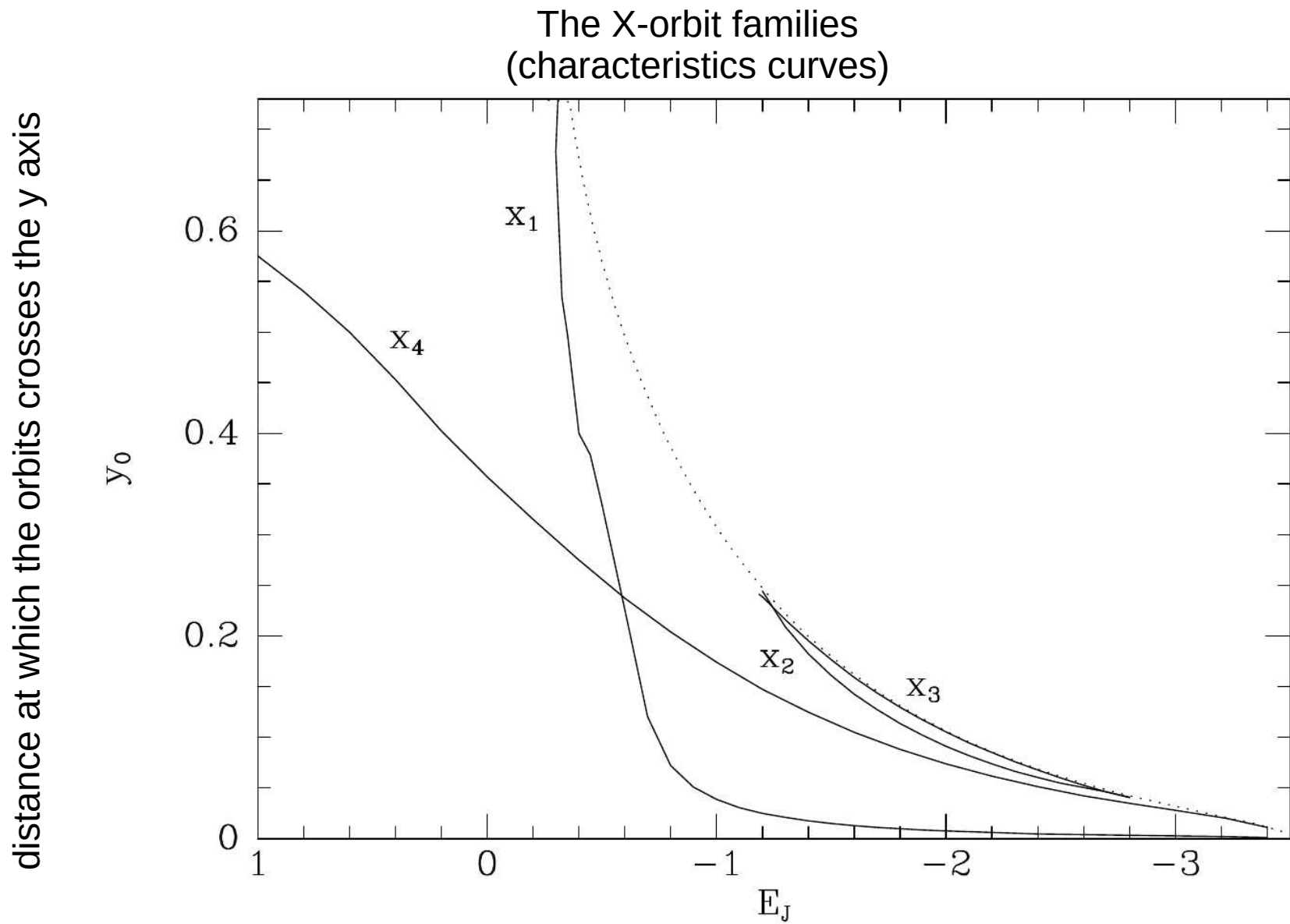


Figure 3.18 A plot of the Jacobi constant E_J of closed orbits in $\Phi_L(q = 0.8, R_c = 0.03, \Omega_b = 1)$ against the value of y at which the orbit cuts the potential's short axis. The dotted curve shows the relation $\Phi_{\text{eff}}(0, y) = E_J$. The families of orbits x_1 – x_4 are marked.

The End