Problem set 9: Solutions

Problem 1

We consider a dark matter detector that measures the recoil energy of a nucleus A of mass M_A, initially at rest, after an elastic collision with a dark matter particle χ, of mass M_χ. The detector has a detection threshold corresponding to a minimal kinetic energy of the nucleus A after the collision.

We make the following assumptions:

- *the elastic collisions are central (head-on collision), such that the initial and final momenta are all parallel;*
- *the typical velocity of the χ particles in our region of the galaxy is approximately similar to the velocity of the sun in the galaxy ($\approx 240\,\mathrm{km/s}$);*
- *the mass per unit volume, $\rho = M_\chi \cdot n$, of the χ particles is a constant, where n is the number of particles per unit of volume.*

(a) *What is the expression for the initial momentum of the dark matter particle? (hint: $\beta = \frac{v}{c} = \frac{p}{E}$ and $\gamma = (1-\beta^2)^{-1/2} = \frac{E}{M}$). Show that in the limit $\beta \ll 1$ we have $p_\chi \simeq \beta M_\chi$. Justify the approximation $\beta \ll 1$.*

(b) *Using energy and momentum conservation, show that the momentum p'_A of the nucleus A after the collision is*

$$p'_A \approx 2\beta\gamma \frac{M_A M_\chi}{M_A + M_\chi},$$

where we used the fact that $p_i \ll M_i$, $i = \chi, A$. Infer from this result that the probability for observing a collision decreases for small masses M_χ, and consequently that the sensitivity of the detector is low for $M_\chi \ll M_A$.

(c) *Find an expression for the number of collisions $N_{\text{collisions}}$ observed during a time Δt, as a function of the cross section and of the density of dark matter particles. Show that for a constant mass density of dark matter, the number of collisions decreases when M_χ increases.*

Solution:

(a) Given that $\beta = \frac{p}{E}$ and $\gamma = \frac{E}{M}$, we simply have:

$$\beta\gamma M = \frac{p}{E} \cdot \frac{E}{M} \cdot M = p\,.$$

Alternatively, using the relation between β and the kinematic quantities, p_χ and E_χ, we have:

$$\beta = \frac{p_\chi}{E_\chi} \Rightarrow p_\chi = \beta E_\chi = \beta\sqrt{M_\chi^2 + p_\chi^2} \Rightarrow p_\chi^2 = \beta^2(M_\chi^2 + p_\chi^2).$$

Then we can write:

$$p_\chi = \frac{\beta}{\sqrt{1-\beta^2}} M_\chi = \beta\gamma M_\chi.$$

We know from the experimental conditions that the velocity is small:

$$\beta = \frac{v}{c} = \frac{240 \times 10^3 [ms^{-1}]}{3 \times 10^8 [ms^{-1}]} = 8 \times 10^{-4} \ll 1.$$

With the approximation $\beta \ll 1$, we have $\gamma \approx 1$, and $p_\chi \simeq \beta M_\chi$.

(b) Considering a one dimensional system, momentum conservation implies:

$$p_\chi = p'_A + p'_\chi\,,$$

and energy conservation gives:

$$E_\chi + E_A = E'_\chi + E'_A\,.$$

Combining these equations, we find:

$$\sqrt{M_\chi^2 + p_\chi^2} + M_A = \sqrt{M_\chi^2 + (p_\chi - p'_A)^2} + \sqrt{M_A^2 + p_A'^2}.$$

Since the momenta are small with respect to the masses, we use the approximation $\sqrt{1+x^2} \simeq 1 + \frac{1}{2}x^2$. By simplifying the above expression we obtain a quadratic equation for which one solution is trivial and the second one is given by:

$$p'_A \approx \frac{2M_A}{M_A + M_\chi} p_\chi.$$

Using the result from (a), we find:

$$p'_A \approx \frac{2M_A M_\chi}{M_A + M_\chi} \beta\gamma.$$

This function of the mass M_χ monotonically increases with M_χ. We conclude that for small values of M_χ, the momentum and the kinetic energy of the nucleus A are small, and the detection efficiency is reduced because of the detection threshold.

(c) The number of collisions is proportional to the particles density n, the velocity v, the cross section σ, and to the integration time of the measurement Δt:

$$N_{\text{collisions}} \propto n \cdot \langle v\sigma \rangle \Delta t.$$

Since the mass of dark matter per unit volume is constant, $M_\chi \cdot n = \text{cst}$, we obtain:

$$N_{\text{collisions}} \propto \frac{\langle v\sigma \rangle}{M_\chi} \Delta t.$$

So the number of collisions decreases when M_χ increases assuming a constant mass density of dark matter.

Problem 2

The superCDMS detector observes 11 *candidate collisions for dark matter particles* χ*, while a background of* 6 ± 1 *events is expected. The exposure time* T *is* 577 *kg-days for germanium detectors with molar mass* $72.64\,\mathrm{g/mol}$*. The mean density of dark matter in the region of the solar system is measured to be* $\rho = (0.39 \pm 0.03)\,[\mathrm{GeV/c^2}]/\mathrm{cm}^3$*, and the velocity of the solar system in the galaxy is* $240\,\mathrm{km/s}$*.*

Determine the cross section of the dark matter particle χ *in Germanium,* σ_χ*, for* $M_\chi = 10\,\mathrm{GeV/c^2}$.

Solution:

The number of candidate dark matter collisions is:

$$\begin{aligned} N &= (N_{\mathrm{obs}} - N_{\mathrm{bkg}}) \pm \sqrt{\sigma^2_{\mathrm{obs}} + \sigma^2_{\mathrm{bkg}} + \sigma^2_{\mathrm{sys.bkg}}} \\ &= (11 - 6) \pm \sqrt{11 + 6 + 1} = 5 \pm 4\,, \end{aligned}$$

where N_{obs} is the number of observed candidate events and N_{bkg} is the expected average number of background events. We added in quadrature the poissonian uncertainties $\sigma_{\mathrm{obs}} = \sqrt{11}$ for N_{obs} and $\sigma_{\mathrm{bkg}} = \sqrt{6}$ for N_{bkg}, plus $\sigma_{\mathrm{sys.bkg}} = 1$ the estimated uncertainty on the background (which may come from theoretical models used for the background estimation, numerical simulations etc.). The number of observed collisions is related to basic quantities as

$$N = n_\chi N_{\mathrm{N}} \sigma_\chi v t\,,$$

where n_χ is the number of χ particles per unit volume, N_{N} the number of nucleons in the detector target, v is the mean velocity for χ particles with respect to the detector, and t is the time of the exposure.

We also have the relations

$$n_\chi = \frac{\rho_\chi}{M_\chi},$$

and

$$N_{\mathrm{N}} t = \frac{T \times N_{\mathrm{Av}}}{m_{\mathrm{molar}}}.$$

Plugging into N the aforementioned relations and expressing it in GeV, seconds, meters, and kilograms, we obtain

$$N = \rho_\chi \frac{T \times 86400 \times N_{\mathrm{Av}}}{m_{\mathrm{molar}}} \times v \times \frac{\sigma_\chi}{M_\chi},$$

and using the numerical values we find

$$N = 0.39 \times 10^6 \times \frac{577 \times 86400 \times 6.022 \times 10^{23}}{0.07264} \times 240 \times 10^3 \times \frac{\sigma_\chi}{M_\chi} = 3.9 \times 10^{43} \times \frac{\sigma_\chi}{M_\chi}\,.$$

So, the cross section for $M_\chi = 10\,\mathrm{GeV/c^2}$ is equal to

$$\sigma_\chi = \frac{5}{3.9 \times 10^{43}} 10 = 1.3 \times 10^{-42}\,\mathrm{m}^2.$$