Problem set 8: Solutions

Problem 1

Considering the scale factor $a(t)$ for the expansion of the Universe, and assuming a flat Universe:

(a) *Determine the time dependence for $a(t)$, assumed to be proportional to a power α of time, $a(t) \propto t^\alpha$, in the case of a Universe containing only radiation. Consider the radiation density to vary as a function of a as $\rho_R \propto a^{-4}$.*

(b) *Determine the expression of $a(t)$ for a matter-dominated Universe (baryonic and dark matter), knowing that the density of matter varies as $\rho_M \propto a^{-3}$.*

(c) *Determine $a(t)$ for the dark energy Λ, for which the density is constant ($\rho_\Lambda = \mathrm{cst}$).*

(d) *Use the above results to explain the evolution of the composition of the Universe (see Fig. 1)?*

Solution:
Assuming a flat Universe ($k = 0$), we use

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 \propto \rho.$$

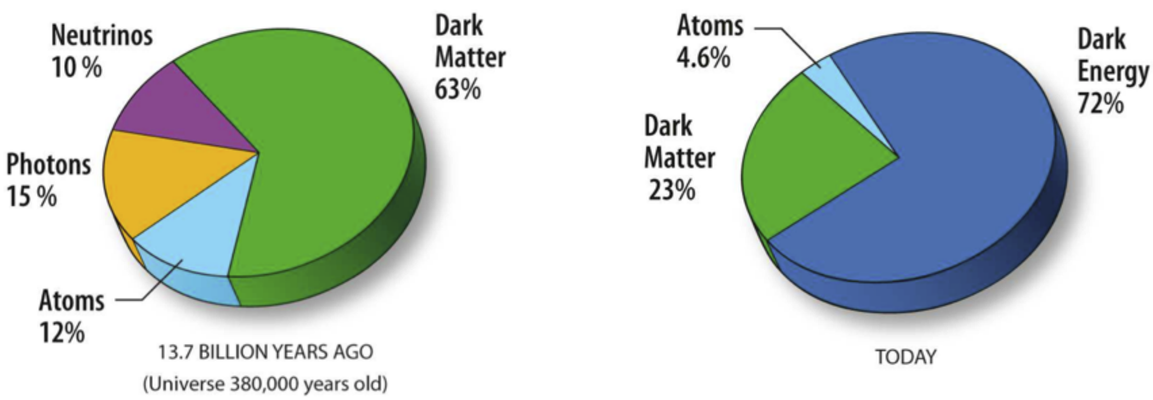

Figure 1: *Composition of the Universe* 13.7 *billion years ago and today.*

If ρ is proportional to the n^{th} power of a, we can write

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^n.$$

We then obtain

$$\dot{a} \propto a^{1+\frac{n}{2}}.$$

If $a(t) \propto t^m$, the solution is $m = -2/n$, only for $n > 0$. In the case where $n = 0$, we find $a(t) \propto e^t$.

(a) For the pure radiation case $n = -4$, and $a(t) \propto t^{1/2}$.

(b) For matter $n = -3$, and then $a(t) \propto t^{2/3}$.

(c) Finally, for dark energy, $a(t) \propto e^t$.

(d) The solutions found for $a(t)$ are valid only for a Universe containing one type of energy. The time evolution of the density of such a Universe is given by $\rho(t) = \rho(a(t))$. However, our Universe contains the three types of components. The time evolution of the density of each component, ρ_i, cannot be directly computed from the solutions for $a_i(t)$.

But we notice that the scale factor $a(t)$ has only increased with time. Therefore, reducing the scale factor a is equivalent to going back in time. Figure 2 shows the density as a function of the scale factor. At low scale factor, respectively at short time after the Big Bang, the Universe was dominated by radiation and the scale factor evolved as $a(t) \propto t^{1/2}$. At large a, the universe is dominated by dark energy and $a(t) \propto e^t$. In between, there is a time where matter dominates, as 380'000 years after the Big Bang, and the scale factor increased as $a(t) \propto t^{2/3}$. Today, the composition of the universe is dominated by dark energy and its expansion rate is exponential.

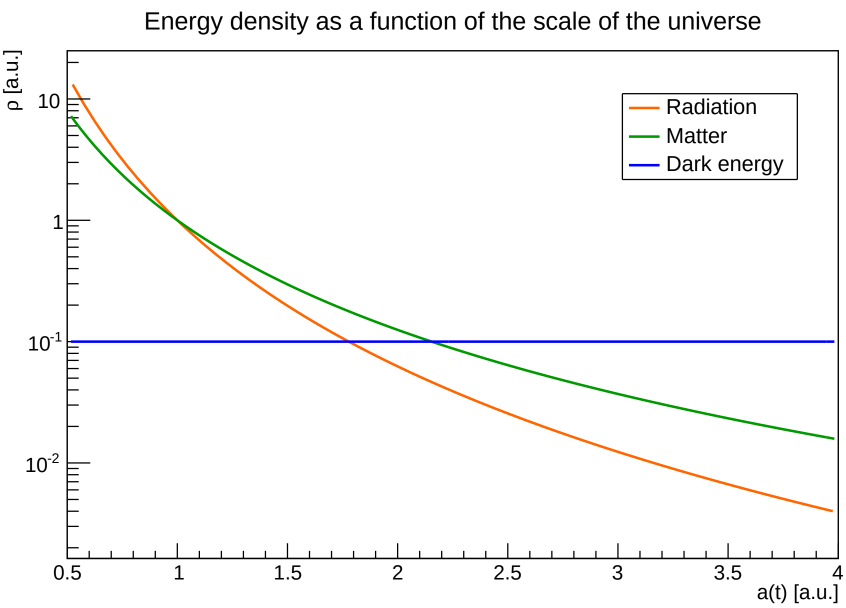

Figure 2: *Evolution of the density of each energy component with the scale of the universe.*