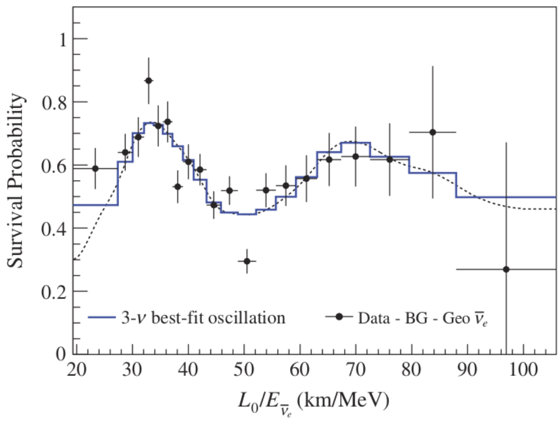
Problem set 5: Solutions

Problem 1

The KamLAND experiment measured the disappearance of $\bar{\nu}_e$ anti-neutrinos produced in reactors located at ~ 180 km from the detector. Estimate the parameter Δm^2_{12} from the plot of the survival probability as a function of L/E. Compare your result with the value determined by the KamLAND experiment: $\Delta m^2_{12} = (7.53 \pm 0.18) \times 10^{-5}\,\mathrm{eV}^2$.



Solution:

The oscillation probability P_{osc} is given by the formula

$$P_{\text{osc}} = \sin^2 2\theta \sin^2\left(\frac{1.27\Delta m^2 L}{E}\right),$$

where the oscillation period is π. Then, considering the equality

$$\frac{1.27\Delta m^2 L}{E} = \pi,$$

we obtain

$$\Delta m^2 = \frac{\pi}{1.27(L/E)_{period}},$$

In order to estimate the period, we measure the separation between the first minimum and the second maximum on the KamLAND plot and multiply it by two. We estimate that the uncertainty is given by the bin width. This leads to $(L/E)_{period} \simeq (40 \pm 4)\,\mathrm{km\,MeV}^{-1}$. Then:

$$\Delta m^2 = (6.2 \pm 0.6) \times 10^{-5}\mathrm{eV}^2.$$

This result is almost compatible with the value stated by KamLAND. On the plot from KamLAND, it is clear that the fit function is not a pure sine squared, as it is distorted by the distribution of distances to reactors and the energy spectrum. The oscillation period can therefore be more precisely measured by accounting for these effects.

Problem 2

Show that the mass–squared eigenvalues for the hamiltonian describing neutrino oscillations in matter, expressed in the (ν_e, ν_μ) basis, are

$$m^2 = \frac{1}{2}(\mu^2 + B) \pm \frac{1}{2}\sqrt{(\Delta m^2 \cos 2\theta - B)^2 + (\Delta m^2 \sin 2\theta)^2}\,,$$

where $\Delta m^2 = m_2^2 - m_1^2$, and that the corresponding mixing angle is given by

$$\tan 2\theta_{\text{matter}} = \frac{\sin 2\theta}{\cos 2\theta - \frac{B}{\Delta m^2}}\,,$$

where $B = 2EV_W$ accounts for the charged-current interaction potential.

Solution:

In order to find the mass–squared eigenvalues for the hamiltonian describing neutrino oscillations in matter, we diagonalise the matrix H_ν, containing an additional potential which is diagonal in the (ν_e, ν_μ) basis. Then the matrix becomes

$$\frac{1}{2E}\begin{pmatrix} M_{ee}^2 + B & M_{\mu e}^2 \\ M_{\mu e}^2 & M_{\mu\mu}^2 \end{pmatrix},$$

where we use the following expressions for the matrix elements

$$M_{ee}^2 = \frac{1}{2}(\mu^2 - \Delta m^2 \cos 2\theta),$$

$$M_{\mu\mu}^2 = \frac{1}{2}(\mu^2 + \Delta m^2 \cos 2\theta),$$

$$M_{e\mu}^2 = \frac{1}{2}\Delta m^2 \sin 2\theta.$$

Solving the eigenvalue problem

$$\left[\frac{1}{2}(\mu^2 - \Delta m^2 \cos 2\theta) + B - m^2\right]\left[\frac{1}{2}(\mu^2 + \Delta m^2 \cos 2\theta) - m^2\right] - \frac{1}{4}(\Delta m^2 \sin 2\theta)^2 = 0,$$

we find

$$m^2 = \frac{\mu^2 + B}{2} \pm \frac{\Delta m^2}{2}\sqrt{\left(\cos 2\theta - \frac{B}{\Delta m^2}\right)^2 + \sin^2 2\theta} = \frac{1}{2}(\mu^2 + B \pm \Delta m^2_{matter})\,. \tag{1}$$

We see that in the limit $B \to 0$ (vacuum), Eq.1 can be rewritten as

$$m^2_{B\to 0} = \frac{\mu^2}{2} \pm \frac{\Delta m^2}{2}\sqrt{\cos^2 2\theta + \sin^2 2\theta} = \frac{1}{2}\left(\mu^2 \pm \Delta m^2\right)\,. \tag{2}$$

By comparing Eqs.1 and 2, we can write:

$$\Delta m^2_{matter} = \Delta m^2 \sqrt{\left(\cos 2\theta - \frac{B}{\Delta m^2}\right)^2 + \sin^2 2\theta}.$$

Replacing this expression in Eq.1, we get:

$$m^2 = \frac{\mu^2 + B}{2} \pm \frac{\Delta m^2_{mater}}{2} \sqrt{\frac{\left(\cos 2\theta - \frac{B}{\Delta m^2}\right)^2}{\left(\cos 2\theta - \frac{B}{\Delta m^2}\right)^2 + \sin^2 2\theta} + \frac{\sin^2 2\theta}{\left(\cos 2\theta - \frac{B}{\Delta m^2}\right)^2 + \sin^2 2\theta}}. \quad (3)$$

Now, if we compare Eq. 3 and the similar expression in vacuum (Eq. 2), we can write:

$$m^2 = \frac{\mu^2 + B}{2} \pm \frac{\Delta m^2_{matter}}{2} \sqrt{\sin^2 2\theta_m + \cos^2 2\theta_m}, \quad (4)$$

where θ_m is the mixing parameter in matter.

Finally, comparing Eqs. 3 and 4, we obtain:

$$\cos 2\theta_m = \frac{\cos 2\theta - \frac{B}{\Delta m^2}}{\sqrt{\left(\cos 2\theta - \frac{B}{\Delta m^2}\right)^2 + \sin^2 2\theta}}$$

$$\sin 2\theta_m = \frac{\sin 2\theta}{\sqrt{\left(\cos 2\theta - \frac{B}{\Delta m^2}\right)^2 + \sin^2 2\theta}}.$$

From which we obtain:

$$\tan 2\theta_m = \frac{\sin 2\theta_m}{\cos 2\theta_m} = \frac{\sin 2\theta}{\cos 2\theta - \frac{B}{\Delta m^2}}.$$