Problem set 4: Solutions

Problem 1

In the two-neutrino system (ν_e and ν_μ), show that the oscillation probability $\nu_e \to \nu_\mu$ is:

$$P_{\text{osc}} = |\langle \nu_\mu | \nu(t) \rangle|^2 = 4 \sin^2\theta \cos^2\theta \sin^2 \frac{(E_1 - E_2)t}{2}$$
$$= \sin^2 2\theta \sin^2 \frac{\Delta m^2\, t}{4E}.$$

Solution:

The oscillation probability P_{osc} for the transition $\nu_e \to \nu_\mu$ is the squared magnitude of the amplitude $\mathcal{A}$, which is given by

$$\mathcal{A} = \langle \nu_\mu | \nu(t) \rangle = -\sin\theta\cos\theta e^{-iE_1 t} + \sin\theta\cos\theta e^{-iE_2 t} = \sin\theta\cos\theta \left(e^{-iE_2 t} - e^{-iE_1 t}\right).$$

For P_{osc}, we obtain

$$\begin{aligned} P_{\text{osc}} &= |\langle \nu_\mu | \nu(t) \rangle|^2 = \langle \nu_\mu | \nu(t) \rangle \cdot \overline{\langle \nu_\mu | \nu(t) \rangle} \\ &= \sin^2\theta\cos^2\theta \left(e^{-iE_2 t} - e^{-iE_1 t}\right)\left(e^{iE_2 t} - e^{iE_1 t}\right) \\ &= \sin^2\theta\cos^2\theta \left[2 - e^{i(E_1 - E_2)t} - e^{-i(E_1 - E_2)t}\right] \\ &= \sin^2\theta\cos^2\theta \left[2 - 2\cos\left((E_1 - E_2)t\right)\right] \\ &= 4\sin^2\theta\cos^2\theta \sin^2 \frac{(E_1 - E_2)t}{2}, \end{aligned}$$

where we used the trigonometric relation $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$.

In order to find the expression for the probability, we consider that the neutrino mass is much smaller than its momentum. Therefore $E_1 \approx E_2 \approx E \approx p \approx p_1 \approx p_2$

$$E_i = \sqrt{p_i^2 + m_i^2} \approx p_i + \frac{m_i^2}{2p_i} \approx E + \frac{m_i^2}{2E}.$$

We can then rewrite

$$P_{\text{osc}} = \sin^2 2\theta \sin^2 \frac{m_1^2 - m_2^2}{4E} t = \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 t}{4E}\right).$$

Problem 2

The mean energy loss of a muon in a medium can be expressed as:

$$\left\langle \frac{dE}{dX} \right\rangle = -(a + bE).$$

The parameter a takes into account the energy loss due to ionization and excitation effects, while parameter b describes Bremsstrahlung, e^+e^- pair production and photo-nuclear reactions. We assume a and b to be constant.
Derive the expression for the mean free path of a muon of energy E_μ in rock. We consider that the particle is stopped for an energy below a certain threshold E_{min}.
For the computation, use: $a \equiv 2\text{MeV}\,\text{cm}^2/\text{g}$, $b \equiv 4 \times 10^{-6}\text{cm}^2/\text{g}$ (for rock as medium), $E_{min} \equiv 1\ GeV$, $E_\mu = 100\ TeV$, and $\rho_{\text{rock}} = 3000\,\text{kg/m}^3$.

Solution:

The mean free path, normalised to the matter density, can be written as:

$$X_\mu = \int_0^{X_\mu} dX = \int_{E_\mu}^{E_{\min}} \frac{dE}{\langle dE/dX \rangle} = \frac{1}{b} \ln\left(\frac{a + bE_\mu}{a + bE_{\min}} \right)$$

The mean free path in rock is $\lambda_\mu = X_\mu/\rho_{\text{rock}} = 4.4\,\text{km}$. This result tells us that the effective detection volume for muon neutrinos is larger than the size of the detector itself, because muons produced by neutrino interactions outside the detector (within a typical distance equal to X_μ) can be detected. For electrons, the mean free path is much shorter, so only neutrinos interacting in, or very near, the detector are observed.