Problem set 3: Solutions

Problem 1

The net reaction in the pp cycle is $4p + 2e^- \rightarrow {}^4\mathrm{He}^{++} + 2\nu_e + Q$. *Knowing that the binding energy* ($\mathcal{B}$) *for* ^{4}He *is* 28.3 *MeV, show that the Q-value of the reaction (energy produced in the process) is* 26.7 *MeV.*

Given the solar luminosity $L_\odot = 3.828 \times 10^{26}\,W$ *and the Earth orbit radius of* 150×10^9 *m, compute the flux of solar neutrinos on Earth.*

Solution:

$$Q = 4m_p + 2m_e - (2m_p + 2m_n - \mathcal{B}(4,2)) = 2m_p + 2m_e - 2m_n + \mathcal{B}(4,2) = 26.7\,\mathrm{MeV} \tag{1}$$

Is the energy produced by the reaction. Knowing the luminosity, *i.e.* the energy emitted per second, of the Sun one can deduce the number of solar neutrinos produced per second.

$$\begin{aligned} L_\odot &= 3.828 \times 10^{26}\ \mathrm{W} = 3.828 \times 10^{26}\ \frac{\mathrm{J}}{\mathrm{s}} \\ &= \frac{3.828 \times 10^{26}}{1.602 \times 10^{-19}}\ \frac{\mathrm{eV}}{\mathrm{s}} \\ &= 2.4 \times 10^{45}\ \frac{\mathrm{eV}}{\mathrm{s}} \end{aligned} \tag{2}$$

Defining N_{pp} as the number of pp cycles we have that:

$$L_\odot = \frac{N_{pp} Q}{\Delta t} = \frac{N_\nu Q}{2\Delta t}, \tag{3}$$

since 2 neutrinos are emitted per pp cycle. The number of neutrinos emitted per second is then:

$$\frac{N_\nu}{\Delta t} = \frac{2}{Q} \cdot L_\odot = \frac{2 \cdot 2.4 \times 10^{45}}{26.7 \times 10^6}\ \mathrm{s}^{-1} = 1.8 \times 10^{38}\ \mathrm{s}^{-1} \tag{4}$$

Assuming the emission of solar neutrinos is isotropic, the flux of solar neutrinos ϕ_ν on Earth is given by:

$$S_{Earth} = 4\pi \cdot (150 \times 10^9\ \mathrm{m})^2 = 2.8 \times 10^{23}\ \mathrm{m}^2 \tag{5}$$

$$\begin{aligned} \phi_\nu = \frac{N_\nu}{\Delta t\ S_{Earth}} = \frac{1.8}{2.8} \times 10^{15}\ \mathrm{m}^{-2}\ \mathrm{s}^{-1} &= 0.64 \times 10^{11}\ \mathrm{m}^{-2}\ \mathrm{s}^{-1} \\ &\approx 6 \times 10^{10}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1} \end{aligned} \tag{6}$$

Problem 2

Compute the minimum energy ("GZK cutoff") at which a proton interacting with the cosmic microwave background (CMB) at 2.725 *K can produce pions in the reaction:*

$$p + \gamma_{\text{CMB}} \to n + \pi^+$$

Solution:

The minimum energy that the proton needs to have in order for the reaction to be possible is the one in which the final state particles are produced at rest in the centre-of-mass frame.

In the centre-of-mass frame, the 4-momenta of the particles are given by:

$$P_p^* = (E_p^*, \vec{p}_p^*),\ P_\gamma^* = (E_\gamma^*, E_\gamma^*),\ P_n^* = (M_n, \vec{0}),\ P_\pi^* = (M_\pi, \vec{0}), \tag{7}$$

and in the lab frame they are given by:

$$P_p^{\text{min}} = (E_p^{\text{min}}, \vec{p}_p^{\text{min}}),\ P_\gamma = (E_\gamma, \vec{p}_\gamma),\ P_n = (E_n, \vec{p}_n),\ P_\pi = (E_\pi, \vec{p}_\pi). \tag{8}$$

Since the CMB is black-body radiation at a temperature $T = 2.725\,\text{K}$, its energy in the lab frame, E_γ, can be computed using Wien's displacement law:

$$\lambda = \frac{2.9 \times 10^{-3}[\text{m K}]}{T} \approx 1.1 \times 10^{-3}\,\text{m}, \tag{9}$$

$$E_\gamma = \frac{hc}{\lambda} = \frac{6.626 \times 10^{-34}[\text{J s}] \cdot 3 \times 10^8[\text{m s}^{-1}]}{1.1 \times 10^{-3}[\text{m}] \cdot 1.602 \times 10^{-19}[\text{J eV}^{-1}]} \approx 11.3 \times 10^{-4}\,\text{eV}. \tag{10}$$

We can now use the Lorentz invariant s to get E_p^{min}:

$$s = (P_p^{\text{min}} + P_\gamma)^2 = (P_n^* + P_\pi^*)^2 \tag{11}$$

$$\Leftrightarrow (P_p^{\text{min}})^2 + 2P_p^{\text{min}} P_\gamma + P_\gamma^2 = P_n^{*2} + P_\pi^{*2} + 2P_n^* P_\pi^* \tag{12}$$

$$\Leftrightarrow M_p^2 + 2E_p^{\text{min}} E_\gamma - 2\vec{p}_p^{\text{min}} \cdot \vec{p}_\gamma = (M_n + M_\pi)^2 \tag{13}$$

The only unknown term is $\vec{p}_p^{\text{min}} \cdot \vec{p}_\gamma$. Since the available energy in a collision, $\sqrt{s}$, (centre-of-mass energy) is maximum when the initial particles collide frontally, in order to make the reaction possible with the minimum proton momentum, the proton and the photon have to collide frontally ($\theta = \pi$). We therefore have:

$$\vec{p}_p^{\text{min}} \cdot \vec{p}_\gamma = -p_p^{\text{min}} p_\gamma = -p_p^{\text{min}} E_\gamma \approx -E_p^{\text{min}} E_\gamma, \tag{14}$$

where we have used $p_p^{\text{min}} \approx E_p^{\text{min}}$ as the proton is ultra-relativistic.

Solving for E_p^{min} we find:

$$E_p^{\text{min}} \approx \frac{(M_n + M_\pi)^2 - M_p^2}{4E_\gamma} = \frac{(939.57 + 139.6)^2 - 938.27^2}{4 \cdot 11.3 \times 10^{-10}} \approx 6.3 \times 10^{13}\,\text{MeV} = 6.3 \times 10^{19}\,\text{eV}.$$

Problem 3

Assuming a total absorption cross-section of $\sigma \approx 10^{-44}\, cm^2$ for neutrinos in matter, constant with neutrino energy.

(a) *Determine the thickness of a wall of lead ($\rho = 11\, g/cm^3$) able to reduce the flux of a beam of neutrinos by 50%.*

(b) *Estimate how many solar neutrinos are absorbed in your body every day, knowing that the flux of solar neutrinos from the pp cycle is $6 \times 10^{10}\, cm^{-2}\, s^{-1}$.*

Solution:

(a) The cross-section σ is a measure of the scattering (here: absorption) probability. One neutrino that passes through a material of number density (number of scatterers per unit volume) n and length ℓ is absorbed with a probability of $n\sigma\ell$. When N_ν neutrinos pass through the material, the expected number of absorbed neutrinos is $N_\nu n\sigma\ell$, which is to say that the number of neutrinos is reduced by $N_\nu n\sigma\ell$. In differential form, one can write $dN_\nu = -N_\nu n\sigma dz$. Since the neutrino flux ϕ is proportional to N_ν, we have:

$$\frac{d\phi}{dz} = -n \cdot \sigma \cdot \phi \tag{15}$$

The nucleon number density n is in a good approximation given by $n = \rho/m_p$ with ρ the density and m_p the proton mass. The solution of the differential equation (15) is:

$$\phi(z) = \phi_0 e^{-n\sigma z}$$

where the product $n \cdot \sigma$ can be interpreted as the differential probability of absorption dP_{abs}/dz. The distance after which half of the neutrinos are absorbed is:

$$z_{1/2} = \frac{\ln(2)}{n \cdot \sigma} = 11\, \mathrm{ly}.$$

(b) The differential probability of absorption in human body is:

$$\frac{dP_{\mathrm{abs}}}{dz} = n_{\mathrm{H_2O}} \cdot \sigma = \frac{\rho_{\mathrm{H_2O}}}{m_p} \cdot \sigma \tag{16}$$

and the probability of absorption for a depth D ($D << 1/\sigma_n$ = mean free path) is $P_{abs}(D) = dP_{abs}/dz \cdot D$.

The number of neutrinos absorbed during a day (Δt) is:

$$N = \phi \cdot S \cdot \Delta t \cdot P_{\mathrm{abs}}(D)$$

where S is the area exposed to neutrinos. Using Eq. (16) and the fact that $S \cdot D$ is the volume, N can be written as:

$$N = \phi \cdot \Delta t \cdot \frac{M}{m_p} \cdot \sigma \approx 2.2 \text{ neutrinos per day}$$

where M is the mass of a human body.