Problem set 1: Solutions

Problem 1

We consider the two-body decay $A \to B + e^-$ of nucleus A, of mass m_A, into nucleus B, of mass m_B, and an electron (m_e). Determine, in the rest frame of nucleus A, the maximum energy of the electron.

Solution:

Energy conservation implies: $E_A = E_B + E_e$. In the rest frame of the parent particle A, $E_A = m_A$, and the momenta of the daughter particles B and e^- are equal in norm and opposite in direction. So we have $E_B^2 = (E_A - E_e)^2 = m_A^2 - 2m_A E_e + E_e^2$.

We can also write $E_B^2 = m_B^2 + p^2$ and $E_e^2 = m_e^2 + p^2$, from which we deduce that $E_B^2 = m_B^2 + E_e^2 - m_e^2$.

Using the two expressions for E_B^2, we obtain:

$$m_A^2 - 2m_A E_e + E_e^2 = m_B^2 + E_e^2 - m_e^2$$

$$\Rightarrow E_e = \frac{m_A^2 - m_B^2 + m_e^2}{2m_A}$$

Problem 2

Draw the Feynman diagram of the following processes and specify the quantum chromodynamics (QCD) colour currents:

$$n \to p\, e^- \, \overline{\nu}_e \tag{1}$$
$$\pi^- \to \mu^- \, \overline{\nu}_\mu \tag{2}$$
$$\tau^- \to \pi^- \, \pi^+ \, \pi^- \, \nu_\tau \tag{3}$$
$$\overline{\nu}_e \, p \to n\, e^+ \tag{4}$$

Solution:

To draw Feynman diagrams, we make use of the Feynman rules displayed in Figure 1, where q_1 refers to a quark of charge $-1/3$ (d, s, b) and q_2 to a quark of charge $+2/3$ (u, c, t), l is a lepton of charge -1 (e, μ, τ) and ν a neutrino $(\nu_e, \nu_\mu, \nu_\tau)$ which has zero charge. In the following diagrams, we choose time to propagate from left to right. Arrows going forward in time refer to fermions (quarks and leptons) whereas arrows going backward in time refer to antifermions (antiquarks and antileptons).

Just like electric charges in the theory of electromagnetism, the theory of QCD (describing the strong nuclear force) has units of charge called colours. Only quarks possess a colour since they are the only fermions sensitive to the strong nuclear force. Quarks can possess three different colours, green (g), red (r) and blue (r), and antiquarks three different anticolours, antigreen ($\overline{g}$), antired ($\overline{r}$) and antiblue ($\bar{b}$). QCD implies that all hadrons, which are compound states of quarks such as mesons (a quark-antiquark pair) and baryons (three quarks), have an overall null colour charge. For instance, for positive pions (made of an up quark and a antidown quark) the only allowed combinations are $r\overline{r}$, $b\bar{b}$ and $g\overline{g}$, whereas baryons have quarks with different colours (grb), forming a colourless hadron. The interaction between two quarks or between a quark and an antiquark is propagated by a boson called the gluon. There exists eight gluons in the Standard Model (SM) of particle physics:

$$
\begin{aligned}
g_1 &= \bar{b}r \\
g_2 &= \overline{g}r \\
g_3 &= \overline{g}b \\
g_4 &= \overline{r}b \\
g_5 &= \overline{r}g \\
g_6 &= \bar{b}g \\
g_7 &= \frac{1}{\sqrt{2}}(\overline{r}r - \overline{g}g) \\
g_8 &= \frac{1}{\sqrt{6}}(\overline{r}r + \overline{g}g - 2\bar{b}b)
\end{aligned}
$$

We see that the first six gluons are not colourless, suggesting that gluons can carry a non-zero colour charge.

(1) $n \to p\, e^- \overline{\nu}_e$ (Figure 2): This process corresponds to the decay of a neutron, which has a lifetime of $880.2 \pm 1.0\,\text{s}$ (about fifteen minutes). The neutron is made of one up quark (charge $+2/3$) and two down quarks (charge $-1/3$). The proton is made of two up quarks and one down quark. This process needs a down quark to transition to an up quark. This can only be achieved via the weak interaction since it is the only interaction in the SM which allows for a transition modifying the flavour of a quark. Such a transition occurs via the propagation of a virtual negatively charged W boson, leading to the creation of an electron and an antielectronic neutrino ($\overline{\nu}_e$). The remaining up and down quarks are called spectator quarks since they do not intervene in the current process.
We assign each quark with a different colour charge to make both hadrons colourless.

(2) $\pi^- \to \mu^- \overline{\nu}_\mu$ (Figure 3): This process corresponds to the decay of a charged pion, which has a lifetime of $(2.6033 \pm 0.0005) \cdot 10^{-8}\,\text{s}$. The π^- is made of a down quark and an antiup quark. Just like the decay of the neutron, this process is in the realm of the the weak interaction and occurs via the propagation of a virtual W^- boson, leading to the creation of a muon and an antimuonic neutrino ($\overline{\nu}_\mu$). The pion needs to be colourless and we choose to make it $r\overline{r}$ in the drawing where colours are displayed with dark colours and anticolours with light colours ($b\bar{b}$ and $g\overline{g}$ are also possible).

(3) $\tau^- \to \pi^- \pi^+ \pi^- \nu_\tau$ (Figure 4): This process corresponds to the decay of a charged tau lepton, which has a lifetime of $(290.3 \pm 0.5) \cdot 10^{-15}\,\text{s}$. The tau neutrino is created via a $\tau^- \to W^- \nu_\tau$ transition. To create the three pions, QCD is at play via mechanisms involving multiple intermediary gluons. Figure 4 displays only one of the possible diagrams: $d\overline{d}$ and $u\overline{u}$ pairs are created through the propagation of gluons carrying non-zero colour charges. Due to the complex

nature of the theory of perturbative QCD, many diagrams with the same probability of occurring are also possible.

(4) $\overline{\nu}_e p \to n e^+$ (Figure 5): This process corresponds to the interaction of a antielectronic neutrino with matter, or more specifically with the proton of a nucleus. As usual for neutrinos, such a process occurs via the weak interaction. It emits a positron in the final state (which can then be detected with adequate instrumentation), leading to the flavour transition inside the proton $u \to d$, transforming the proton to a neutron. Such a process is very rare and can be observed in the presence of a very high flux of incident neutrinos.

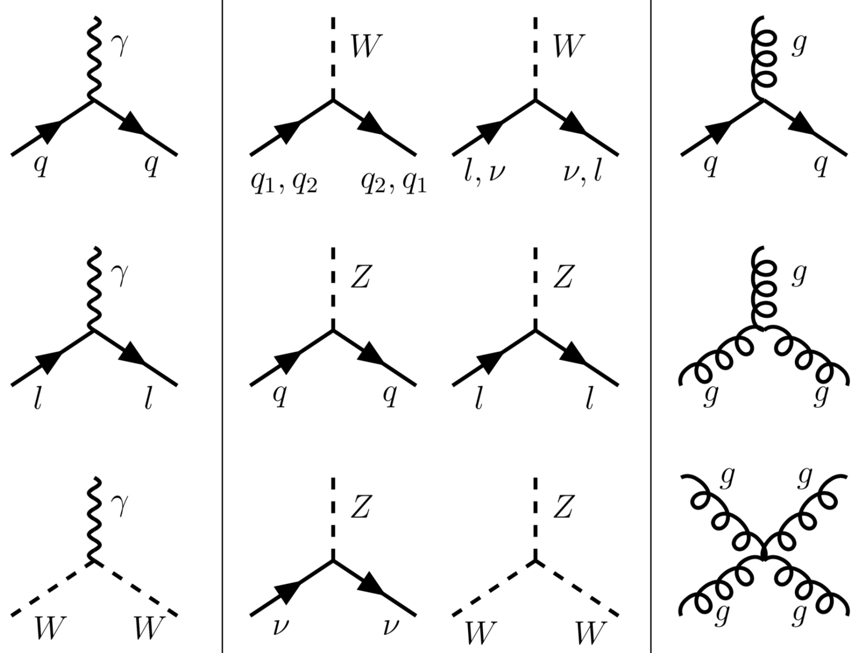
Interaction	électromagnétique	faible	forte
Particules sensibles	chargées q, l, W	quarks et leptons q, l, ν	quarks et gluons q, g
Bosons (spin 1) d'échange	photon γ	W^+, W^-, Z^0	8 gluons g
vertex			

Figure 1: Feynman rules (in french). Taken from Prof. Olivier Schneider's 3rd year course on Particle and Nuclear Physics: `https://lphe.epfl.ch/oschneid/cours/cours3_2003-2004/OS_phys_nucl_corpusc_oct2003_correct.pdf`

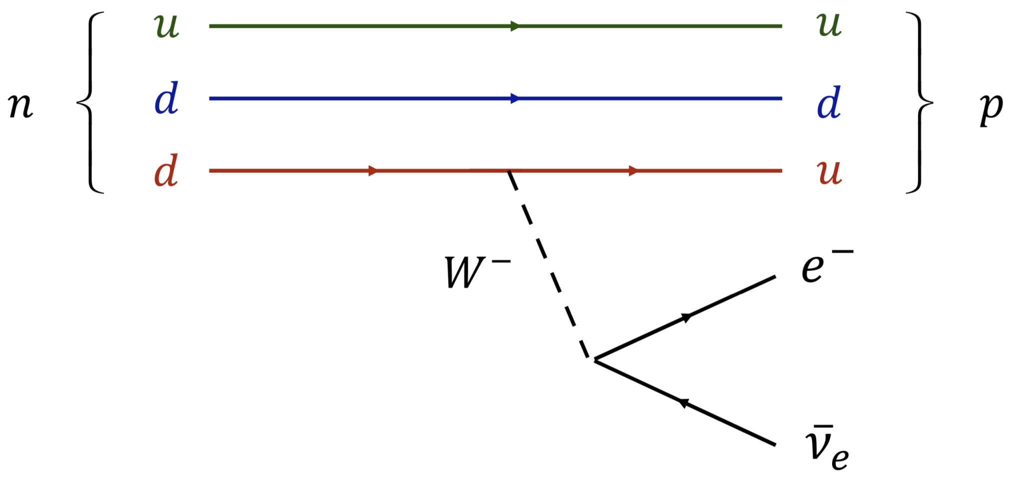


Figure 2: Neutron decay.

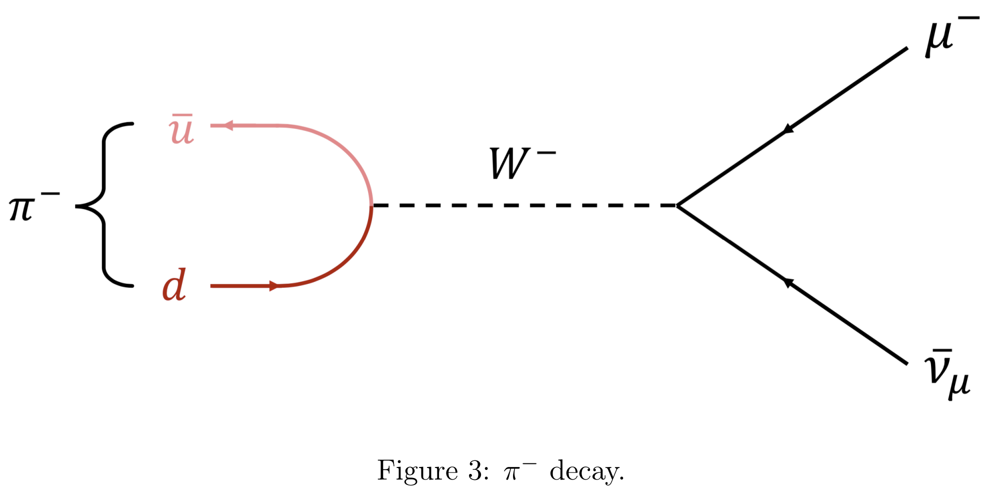


Figure 3: π^- decay.

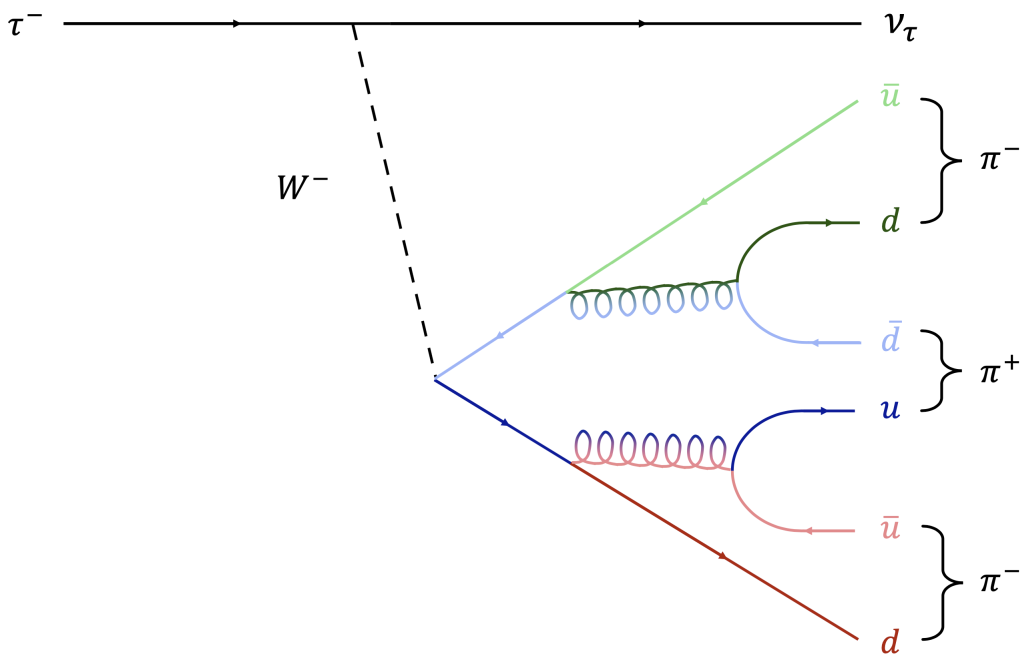


Figure 4: τ^- decay. Light colours correspond to QCD anti-colours while dark colours refer to QCD colours.

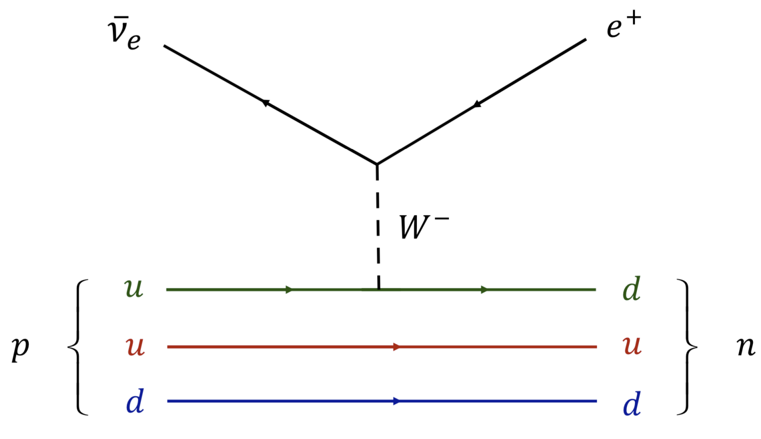


Figure 5: $\overline{\nu}_e$ interaction with matter.

Problem 3

In β decays, show that the fraction of electrons having a kinetic energy T_e in the range ΔT close to the maximum energy Q_β is proportional to $(\Delta T/Q_\beta)^3$. Consider the case $Q_\beta \gg m_e$.

Solution:

We want to calculate the ratio $n(\Delta T)/n_{\text{total}}$. First, we compute:

$$n_{\text{total}} = \int_0^{Q_\beta} \frac{d\Gamma}{dT_e} dT_e \propto \int_0^{Q_\beta} (T_e + m_e)\sqrt{T_e(T_e + 2m_e)}(Q_\beta - T_e)^2 dT_e.$$

Using the approximation $Q_\beta \gg m_e$, we obtain:

$$n_{\text{total}} \propto \int_0^{Q_\beta} T_e^2 (Q_\beta - T_e)^2 dT_e = \frac{1}{3}Q_\beta^2 T_e^3 - \frac{1}{2}Q_\beta T_e^4 + \frac{1}{5}T_e^5 \Big|_0^{Q_\beta} \propto Q_\beta^5. \tag{5}$$

For an interval ΔT close to $T_e = Q_\beta$, we have $T_e \approx Q_\beta$. The number of electrons $n(\Delta T)$ will be:

$$n(\Delta T) = \int_{Q_\beta - \Delta T}^{Q_\beta} \frac{d\Gamma}{dT_e} dT_e \propto \int_{Q_\beta - \Delta T}^{Q_\beta} (Q_\beta + m_e)\sqrt{Q_\beta(Q_\beta + 2m_e)}(Q_\beta - T_e)^2 dT_e.$$

Using again the approximation $Q_\beta \gg m_e$, we find:

$$n(\Delta T) \propto -\frac{1}{3}Q_\beta^2 (Q_\beta - T_e)^3 \Big|_{Q_\beta - \Delta T}^{Q_\beta} \propto Q_\beta^2 (\Delta T)^3. \tag{6}$$

Finally, by combining Eqs. (5) and (6), we demonstrate that:

$$\frac{n(\Delta t)}{n_{\text{total}}} \propto \left(\frac{\Delta T}{Q_\beta}\right)^3$$