Problem set 11: Solutions

Problem 1

Show that the latent heat for the transition of a pions gas into a plasma of quarks and gluons is equal to $4B$, where B is the energy density for the "bag" creation in the bag model which describes quark confinement into hadrons.

Solution:

The critical temperature is:

$$T_c^4 = \frac{90}{37\pi^2}B.$$

At this temperature the energy density for the gas of pions is:

$$\epsilon_\pi(T_c) = \frac{\pi^2}{10}T_c^4 = \frac{9}{37}B,$$

and for the QGP:

$$\epsilon_{\text{QGP}}(T_c) = \frac{40}{30}\pi^2 T_c^4 + B = \frac{120}{37}B + B.$$

The latent heat is given by the difference

$$\Delta\epsilon = \frac{120}{37}B + B - \frac{9}{37}B = 4B.$$

Problem 2

An ideal gas of fermions (bosons) in thermal equilibrium obeys Fermi-Dirac statistics (Bose-Einstein statistics): for each degree of freedom, the average number of particles that occupy a state of energy E is

$$\frac{1}{\exp\left(\frac{E}{k_B T}\right) + 1} \quad \textit{for fermions and} \tag{1}$$

$$\frac{1}{\exp\left(\frac{E}{k_B T}\right) - 1} \quad \textit{for bosons.} \tag{2}$$

Calculate the energy density per degree of freedom for these ideal gases. Note the following identities:

$$\int_0^\infty \frac{x^3}{e^x - 1}\mathrm{d}x = \Gamma(4)\mathrm{Li}_4(1) \tag{3}$$

$$\int_0^\infty \frac{x^3}{e^x + 1}\mathrm{d}x = -\Gamma(4)\mathrm{Li}_4(-1) \tag{4}$$

where $\Gamma(n) = (N)$ is the Gamma function and $\mathrm{Li}_N(x)$ is the polylogarithm. The function $\mathrm{Li}_N(x)$ is related to the Riemann zeta function, $\zeta(N)$, and to the Dirichlet eta function, $\eta(N) = (1-2^{1-N})\zeta(N)$, as

$$\mathrm{Li}_N(1) = \zeta(N)\,, \tag{5}$$

$$\mathrm{Li}_N(-1) = -\eta(N) = -(1-2^{1-N})\zeta(N)\,. \tag{6}$$

And the Riemann zeta function (with $\Gamma(N) = (N-1)!$):

$$\zeta(N) = \frac{1}{\Gamma(N)}\int_0^\infty \frac{u^{N-1}}{e^u-1}\mathrm{d}u \tag{7}$$

$$\Rightarrow \zeta(4) = \frac{1}{6}\int_0^\infty \frac{u^3}{e^u-1}\mathrm{d}u = \frac{\pi^4}{90}\,. \tag{8}$$

Solution:

The average energy per degree of freedom is summed over all possible energy states E_n

$$\sum_n \frac{E_n}{e^{\frac{E_n}{k_BT}} \pm 1}$$

And therefore, the energy density for one degree of freedom is

$$\epsilon = \frac{1}{V}\sum_n \frac{E_n}{e^{\frac{E_n}{k_BT}} \pm 1}$$

For particles of zero mass, we have:

$$E(p) = \sqrt{p^2+m^2} = p \qquad \Rightarrow \qquad \epsilon = \frac{1}{V}\sum_n \frac{p_n}{e^{\frac{p_n}{k_BT}} \pm 1}\,.$$

Instead of summing we integrate over the phase space: $\sum_n = \frac{V}{(2\pi)^3}\int \mathrm{d}^3p$

$$\epsilon = \int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{\frac{p}{k_BT}} \pm 1}\,.$$

With

$$\mathrm{d}^3p = 4\pi p^2 \mathrm{d}p$$

and a change of variable (with $k_B = 1$)

$$x = \frac{p}{T} \qquad \Rightarrow \qquad \mathrm{d}x = \frac{\mathrm{d}p}{T}\,,$$

we can write

$$\epsilon = \frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{p^3}{e^{\frac{p}{T}} \pm 1}\mathrm{d}p = \frac{4\pi}{(2\pi)^3}T^4\int_0^\infty \frac{x^3}{e^x \pm 1}\mathrm{d}x\,.$$

The value of the integral depends on the sign:

$$\int_0^\infty \frac{x^3}{e^x - 1} \mathrm{d}x = \Gamma(4)\mathrm{Li}_4(1) = \Gamma(4)\zeta(4)$$
$$\int_0^\infty \frac{x^3}{e^x + 1} \mathrm{d}x = \Gamma(4)(1 - 2^{1-4})\zeta(4)$$

Finally, we obtain the energy density for bosons

$$\epsilon_{\mathrm{bosons}} = \frac{4\pi}{(2\pi)^3} T^4 \, 6 \, \frac{\pi^4}{90} = \frac{\pi^2}{30} T^4 \,,$$

and for fermions

$$\epsilon_{\mathrm{fermions}} = \frac{4\pi}{(2\pi)^3} T^4 \, 6 \, \underbrace{(1 - 2^{1-4})}_{=7/8} \frac{\pi^4}{90} = \frac{7}{8} \frac{\pi^2}{30} T^4 \,.$$

Carrying all the factors of k_B, $\hbar$ and c yields

$$\epsilon_{\mathrm{bosons}} = \frac{\pi^2}{30} \frac{k_B^4}{\hbar^3 c^3} T^4$$
$$\epsilon_{\mathrm{fermions}} = \frac{7}{8} \frac{\pi^2}{30} \frac{k_B^4}{\hbar^3 c^3} T^4$$