Problem set 10: Solutions

Problem 1

- *Determine the value of Λ_{QCD} for three fermions ($N_f = 3$), knowing that the strong coupling constant has been measured at the Z^0 mass: $\alpha_s(M_Z) = 0.12$.*
- *Compute the strong potential energy (binding energy) for the mesons ϕ ($s\bar{s}$), J/Ψ ($c\bar{c}$) and $\Upsilon(1S)$ ($b\bar{b}$), for which $m_\phi = 1020\,\text{MeV}$, $m_{J/\Psi} = 3100\,\text{MeV}$, and $m_\Upsilon = 9500\,\text{MeV}$. Assume the typical radius of the meson to be of order 1 fm.*

Solution:

For $q = 91.2\,\text{GeV}$, $N_\text{f} = 3$, and $\alpha_s = 0.12$, the coupling constant is given by:

$$\alpha_s(q^2) = \frac{12\pi}{33 - 2N_\text{f}} \frac{1}{\ln \frac{q^2}{\Lambda^2_{\text{QCD}}}}.$$

Solving the equation in terms of Λ_{QCD}, we find:

$$\Lambda_{\text{QCD}} = \left(\frac{q^2}{e^{\frac{12\pi}{(33-2N_\text{f})\alpha_s(q^2)}}} \right)^{\frac{1}{2}} = 0.27\,\text{GeV} = 270\,\text{MeV}.$$

The potential energy is given by:

$$V(r) = -\frac{4}{3}\alpha_s \frac{1}{r} + b \cdot r,$$

where $b = 1\,\text{GeV/fm}$, and $r \approx 1\,\text{fm}$ for the mesons. Then the potential in GeV unit is:

$$V = -\frac{4}{3}\alpha_s \hbar c + 1,$$

where $\hbar c = 0.197\,\text{GeV/fm}$. The coupling constants for each meson are:

$$\alpha_s(m^2_\phi) = \frac{12\pi}{27} \frac{1}{\ln \frac{m^2_\phi}{\Lambda^2_{\text{QCD}}}} = 0.53,$$

$$\alpha_s(m^2_{J/\Psi}) = \frac{12\pi}{27} \frac{1}{\ln \frac{m^2_{J/\Psi}}{\Lambda^2_{\text{QCD}}}} = 0.29,$$

$$\alpha_s(m^2_\Upsilon) = \frac{12\pi}{27} \frac{1}{\ln \frac{m^2_\Upsilon}{\Lambda^2_{\text{QCD}}}} = 0.20.$$

Finally, for each meson, we find the following binding energies between quarks:

$$V(\phi) = 0.87\,\text{GeV},$$
$$V(J/\Psi) = 0.92\,\text{GeV},$$
$$V(\Upsilon) = 0.95\,\text{GeV}.$$

Problem 2

Show that the number of degrees of freedom for

- *all the quarks and gluons is* 79.
- *all the particles of the standard model (quarks, leptons, gauge bosons, Higgs boson) is* 106.75.

Solution:

The number of degrees of freedom (NDF) for each kind of particle is:

- Quarks: 6 flavours (u, d, s, c, b, t), 2 spins ($+1/2$, $-1/2$), 3 colours, 2 states (particle, anti-particle). Then the NDF is $6 \times 2 \times 3 \times 2 = 72$.
- Charged leptons : 3 flavours (e, μ, τ), 2 spins ($+1/2$, $-1/2$), 2 states (particle, anti-particle).The total is $3 \times 2 \times 2 = 12$.
- Neutral leptons (neutrinos): 3 flavors (e, μ, τ), 2 states (particle, anti-particle). The total NDF is $3 \times 2 = 6$. Note: neutrino is always left-handed (spin $-1/2$ along momentum direction), and the anti-neutrino is always right-handed (spin $+1/2$ along momentum direction). There is therefore only one state for the particle spin, and one for the anti-particle.
- The photon has 2 degrees of freedom due to its polarisation (a massless particle cannot be at rest in any frame).
- Gluons: 8 gluons, 2 polarisations (no mass). Then the total is $8 \times 2 = 16$.
- Z^0 has 3 degrees of freedom due to its spin (-1, 0, $+1$).
- For the $W^{\pm}$ we have 3 NDF for the spin (-1, 0, $+1$), and 2 for the charge. The total is $3 \times 2 = 6$.
- Finally, for the Higgs we have only 1 degree of freedom, since its has no charge nor spin.

Then, taking into account the previous considerations:

- For quarks (fermions) and gluons (bosons) the NDF is,

$$\frac{7}{8}72 + 16 = 79.$$

- For all the particles of the standard model the NDF is:

$$\frac{7}{8}(72 + 12 + 6) + (2 + 16 + 3 + 6 + 1) = \frac{7}{8}90 + 28 = 106.75.$$