Gauge invariance in Schrödinger's equation

$$i\hbar\partial_t\psi = \frac{1}{2m}\left[\vec{\hat{p}} - q\vec{A}\right]^2\psi + q\varphi(\vec{r})\psi$$

$$i\hbar\partial_t\psi' = \frac{1}{2m}\left[\vec{\hat{p}} - q\vec{A}'\right]^2\psi' + q\varphi'(\vec{r})\psi'$$

with $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$

$\varphi' = \varphi - \partial_t\Lambda$

$\psi' = e^{iq\Lambda/\hbar}\cdot\psi$

Proof of the equivalence:

Let's start with $i\hbar\partial_t\psi' = i\hbar\,\partial_t\left(e^{iq\Lambda/\hbar}\psi\right)$

$$= i\hbar\left(iq/\hbar\,\partial_t\Lambda\right)e^{iq\Lambda/\hbar}\psi + i\hbar e^{iq\Lambda/\hbar}\partial_t\psi$$

$$= e^{iq\Lambda/\hbar}\left[-q\partial_t\Lambda + i\hbar\partial_t\right]\psi$$

Lemma: $\left[\vec{\hat{p}} - q\vec{A}'\right]\psi' = \left[-i\hbar\vec{\nabla} - q\vec{A}'\right]e^{iq\Lambda/\hbar}\psi$

$$= -i\hbar\left(\frac{iq}{\hbar}\vec{\nabla}\Lambda\right)e^{\frac{iq\Lambda}{\hbar}}\psi - i\hbar e^{iq\Lambda/\hbar}\vec{\nabla}\psi - q\vec{A}'e^{iq\Lambda/\hbar}\psi$$

$$= e^{iq\Lambda/\hbar}\left[-i\hbar\vec{\nabla} \underbrace{- q\vec{A}' + q\vec{\nabla}\Lambda}_{-q\vec{A}}\right]\cdot\psi$$

$$= e^{iq\Lambda/\hbar}\left[\vec{\hat{p}} - q\vec{A}\right]\cdot\psi$$

So we get $\frac{1}{2m}\left(\vec{\hat{p}} - q\vec{A}'\right)^2 e^{iq\Lambda/\hbar}\psi = \frac{1}{2m}\left(\vec{\hat{p}} - q\vec{A}'\right)\cdot e^{iq\Lambda/\hbar}\left(\vec{\hat{p}} - q\vec{A}\right)\psi$

$$= e^{iq\Lambda/\hbar}\cdot\frac{1}{2m}\left(\vec{\hat{p}} - q\vec{A}\right)^2\psi$$

Thus: $i\hbar\partial_t\psi' = \frac{1}{2m}\left(\vec{\hat{p}} - q\vec{A}'\right)\psi' + q\varphi'(r)\psi$

$$\Leftrightarrow\quad e^{i\Lambda q/\hbar}\left[-q\partial_t\Lambda + i\hbar\partial_t\psi\right] = e^{iq\Lambda/\hbar}\frac{1}{2m}\left(\vec{\hat{p}} - q\vec{A}\right)^2\psi + e^{iq\Lambda/\hbar}\left(q\varphi\psi - q\partial_t\Lambda\psi\right)$$

$$\Leftrightarrow\quad i\hbar\partial_t\psi = \frac{1}{2m}\left(\vec{\hat{p}} - q\vec{A}\right)^2 + q\varphi\psi \quad \blacksquare$$