Problem Set 9: Light Propagation

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Exercise 1: The Ray Transfer - ABCD Matrix

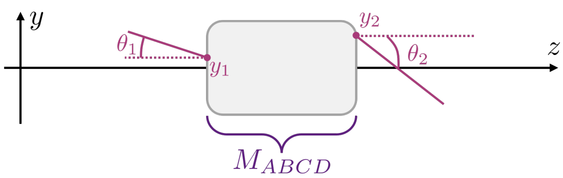
In the framework of geometric optics and under the paraxial approximation, light propagation problems can be addressed using the ABCD matrix model. This is what we propose to study in this first exercise. This tool is particularly useful for understanding the mechanism of resonators, which, in addition to an amplifying medium, constitute the second main component required for the functioning of a laser. Additionally, within the paraxial approximation, this formalism can also be used to predict the propagation of Gaussian beams through various optical systems.

1. Recall Snell-Descartes' laws in the paraxial approximation.

The light beam being described only by its position y and its angle θ, the ABCD matrix M_{ABCD}, or ray transfer matrix, is defined for the propagation of light rays through an optical system as:

$$\begin{pmatrix} y_2 \\ \tilde{\theta}_2 \end{pmatrix} = M_{ABCD} \begin{pmatrix} y_1 \\ \tilde{\theta}_1 \end{pmatrix}, M_{ABCD} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $\tilde{\theta}_{1/2} = n\theta_{1/2}$ are the reduced angles and with $(y_{1,2}, \theta_{1,2})$ defined on the graph below. This matrix has a unitary determinant, $\det(M_{ABCD}) = 1$.



2. Calculate the ABCD matrix for the following optical systems:

- a) Flat interface between two media with respective refractive indexes n_1 and n_2;
- b) A medium with refractive index n and thickness d;
- c) A spherical surface with a radius of curvature R separating two mediums of refractive indexes n_1 and n_2.

The ABCD matrix of an optical system can be decomposed into subsystems as follows:

$$M_{\text{tot}} = M_N \times M_{N-1} \times \ldots \times M_2 \times M_1.$$

3. A thin lens can be considered as two adjacent spherical interfaces with radius of curvature R. The ABCD matrix associated with a thin lens can be written as

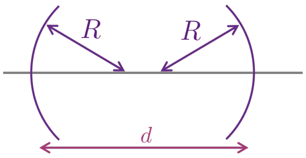
$$M_{ABCD} = \begin{pmatrix} 1 & 0 \\ -n/f & 1 \end{pmatrix}$$

with n the refractive index of the external medium and f the focal length of the lens. Write the focal length f as a function of the refractive index of the medium n, the radius of curvature of the spherical surfaces R and the refractive index n_ℓ of the lens medium.

Application to the stability of resonators

Resonators are optical systems in which light remains trapped, making numerous round trips before escaping. Optical cavities, such as the Fabry-Pérot type, are among the most well-known examples. In addition to the presence of constructive or destructive interference, which allows the selection of specific frequencies of light waves, it is necessary to ensure that the geometry of the resonator allows for the stability of the light rays.

The resonator studied here is a resonator composed of two spherical mirrors with respective radius of curvature R.



4. Calculate the ABCD matrix of a flat mirror.

5. Calculate the ABCD matrix of a spherical mirror with a radius of curvature R. Recover the ABCD matrix for a flat mirror.

6. Express the ABCD matrix M after one round trip in the resonator. What is the ABCD matrix M_N after N round trips ?

7. In general, show that the condition on the coefficients of the ABCD matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of a resonator for it to be considered stable is given by

$$0 \leq \frac{A+D+2}{4} \leq 1$$

Hint: write the matrix in its diagonal form $M = PDP^{-1}$ *with* $D = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ *and find a stability condition on the parameters* $x_{1/2} = |x_{1/2}|\exp(i\varphi_{1/2})$. *Notice that* $\det(M) = 1$.

8. Deduce the condition of stability for the cavity formed by two spherial mirrors.

9. The code below allows you to obtain the position y and the angle θ of a light ray after N passes through a cavity composed of two spherical mirrors with the same radius of curvature R. Verify the stability condition obtained in the previous question.

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

n = 1.0        ## refractive index
d = 1.0        ## resonnator length

################   TO FILL ##################
R = XXX*d      ## radius of curvature
################   TO FILL ##################

y_theta = np.array([0.05*d,np.pi*1.0/180])   ## initial conditions

N = 100
y_array = np.zeros(N)
theta_array = np.zeros(N)
y_array_right = np.zeros(N)
theta_array_right = np.zeros(N)

###### elementary ABCD matrices ########
M1 = np.array([[1,0],[-2*n/R,1]])
M2 = np.array([[1,d/n],[0,1]])
M3 = np.array([[1,0],[-2*n/R,1]])
M4 = np.array([[1,d/n],[0,1]])
############################################

M_ABCD_right = np.dot(M3,M4)
M_ABCD = np.dot(np.dot(M1,M2),np.dot(M3,M4))

for i in range(N):
    M_ABCD_i = np.linalg.matrix_power(M_ABCD, i)
    y_array[i] = np.dot(M_ABCD_i, y_theta)[0]
    theta_array[i] = np.dot(M_ABCD_i, y_theta)[1]
    M_ABCD_right_i = np.dot(M_ABCD_right,M_ABCD_i)
    y_array_right[i] = np.dot(M_ABCD_right_i, y_theta)[0]
    theta_array_right[i] = np.dot(M_ABCD_right_i, y_theta)[1]

##### to draw the mirrors of the cavity ######
def F_cercle_xy (fy, fR, fd):
    return -np.sqrt(fR**2 - fy**2) - fR - fd*0.5

Y_mirror = np.linspace(-5*y_theta[0],5*y_theta[0])
X_array_left = F_cercle_xy(Y_mirror, R,d)
X_array_right = -F_cercle_xy(Y_mirror, R,d)
############################################


##### PLOT : ray propagation through the resonators #####
colors = np.linspace(0.9, 0.2, N-1)
```

```
colormap = plt.cm.magma
plt.rcParams['figure.figsize'] = (12, 5)

plt.figure()
for i in range(N-1):
    if (R**2 - y_array[i]**2 >0) and (R**2 - y_array_right[i]**2 >0) and (R**2 - y_array[i+1]**2 >0) :
        plt.plot(np.array([F_cercle_xy(y_array[i],R,d), -F_cercle_xy(y_array_right[i],R,d)])/d,\
                 np.array([y_array[i], y_array_right[i]])/d,color=colormap(colors[i]))
        plt.plot(np.array([-F_cercle_xy(y_array_right[i],R,d), F_cercle_xy(y_array[i+1],R,d)])/d,\
                 np.array([y_array_right[i], y_array[i+1]])/d,color=colormap(colors[i]))
    else :
        plt.plot(np.array([-0.5*d,0.5*d])/d,np.array([y_array[i], y_array_right[i]])/d,color=colormap(colors[i]))
        plt.plot(np.array([0.5*d,-0.5*d])/d,np.array([y_array_right[i], y_array[i+1]])/d,color=colormap(colors[i]))
plt.plot(X_array_left/d, Y_mirror/d, color = "gray", linewidth = 2)
plt.plot(X_array_right/d, Y_mirror/d, color = "gray", linewidth = 2)
plt.xlabel("x/d")
plt.ylabel("y/d")
plt.show()
##########################################################################
```

Exercise 2: Gaussian Beams and Ray Transfer Matrices

We are interested in a type of wave equation solution in the paraxial approximation which plays a central role in laser physics: the gaussian beams. The fundamental gaussian mode is called the TEM_{00} mode. In one dimension, the field propagates according to

$$\mathcal{E}(x,z) = \mathcal{E}_0 \frac{w_0}{w(z)} e^{i\psi(z)} \exp\left[ik\frac{x^2}{2q(z)}\right]$$

with w_0 the beam waist (in $z=0$) and $q(z)$ the complex radius of curvature defined as

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i\lambda}{\pi w(z)^2}$$

where $R(z)$ is the radius of curvature.

In this context, the ABCD matrix formalism is still valid: the propagation of the Gaussian beam can be described by defining the matrix $M_{ABCD} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, such that

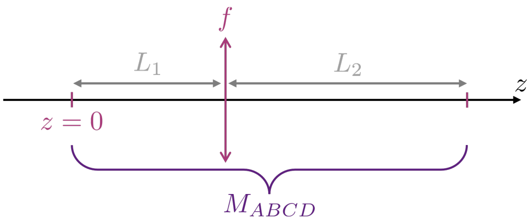
$$q(z_2) = \frac{Aq(z_1)+B}{Cq(z_1)+D}$$

In this exercice, the refractive index is set to $n=1$. We recall that the waist position w_0 of the Gaussian beam lies at $z=0$.

1. Calculate the complex radius of curvature of the Gaussian beam $q(0)$ as a function of the Rayleigh length $z_R = \pi w_0^2/\lambda$.
2. The beam propagates in free space. Deduce the complex radius of curvature at a position z using the matrix M_{ABCD} obtained in the previous exercise.
3. Find the expression for the radius of curvature $R(z)$ and the transverse size $w(z)$ as a function of the waist w_0 and the Rayleigh length z_R.

Determining the waist position of a Gaussian beam as it propagates through a lens

Consider a Gaussian beam of rayleigh length z_{R1} and waist w_1. The position of waist w_1 is at a distance L_1 from a simple converging lens of focal length f. We want to characterize the Gaussian beam at a distance L_2 from the lens.



As in the case of geometrical optics, the matrix describing the propagation of a Gaussian beam through a converging lens of focal length f is written as follows

$$M_{ABCD} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}.$$

4. Calculate the matrix ABCD associated with the optical system.

5. The length L_2 is chosen as corresponding to the waist position of the Gaussian beam. Show that this distance L_2 is related to the position of the initial waist L_1 by the equality

$$d_2 = \frac{\Delta^2 + d_1 \times (d_1 - 1)}{\Delta^2 + (d_1 - 1)^2} \text{ with } d_1 = \frac{L_1}{f}, d_2 = \frac{L_2}{f} \text{ and } \Delta = \frac{z_{R1}}{f}.$$

The code below represents the position L_2 of the waist at the output of the optical system as a function of the position L_1 of the initial waist for different Rayleigh lengths z_{R1}.

6. Discuss the evolution of the curves: make the link with the predictions of geometrical optics, in particular for $L_1 = f$.

In [8]:
```
import numpy as np
import matplotlib.pyplot as plt

def F_conjug (fd1, fDelta):
    return (fDelta**2 + fd1*(fd1-1))/(fDelta**2 + (fd1-1)**2)

Npoints = 1000
D1_array = np.linspace(-3,4, Npoints)
Delta_array = np.array([0.0,0.25,0.5,1.0,2.0])

colors = np.linspace(0.1, 0.9, Delta_array.shape[0])
colormap = plt.cm.magma
plt.rcParams['figure.figsize'] = (7, 5)

plt.figure()
plt.title("Waist position $L_2$ at the output of the lens according to the initial waist position $L_1$")
for i in range(Delta_array.shape[0]):
    plt.plot(D1_array, F_conjug(D1_array, Delta_array[i]), '.', color = colormap(colors[i]), label = "$z_R/f = $"+st
plt.ylim(-3,4)
plt.legend()
plt.xlabel('$L_1 / f$')
plt.ylabel('$L_2 / f$')

plt.grid()
plt.show()
```