Problem Set 8: Classical description of the light matter interaction

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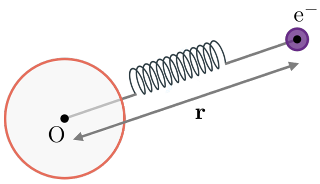
Exercise 1: A classical model, the harmonically bound electron

In this exercise, we present a simple classical approach to describe the atom-fied interaction. We will describe the interaction of a classical field with an harmonically bound charge. This is admittedly a rather naive model of an atom (often called Thomson model, and proposed at the end of the 19th century when the existence of the electron was first revealed). However, it captures many essential aspects of the interaction and its predictions coincide rather well with those of a more refined semi-classical model.

The harmonically bound electron

We consider a single electron of mass m and charge $q = -e$ bounded to the origin (the atom is motionless) by a harmonic force. The position of the electron being $\mathbf{r}$, the motion equation is

$$\ddot{r} + \omega_0^2 r = 0$$



The situation described above is not physical. To make it more physical, we should include some damping to reflect the finite lifetime of the excited states. We will consider here only damping corresponding to the emission of light by the accelerated electron. We assume that the total average power radiated by the electron is given by the Larmor formula

$$\mathcal{P} = \frac{q^2 \mathbf{a}^2}{6\pi\epsilon_0 c^3} = m\tau\mathbf{a}^2$$

with $\mathbf{a}$ the acceleration of the electron and ϵ_0 the vacuum permittivity.

1. Deduce from the power expression that the effective damping force can be written as

$$\mathbf{f}_r = m\tau \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}.$$

Hint : to write radiated power as that of a force, it is possible to do so by integrating the power over a time interval that is long in relation to the electron's oscillation frequency, but short in relation to the timescale of energy dissipation.

2. Assuming that the motion is nearly harmonic, rewrite this damping force as a function of m, τ, ω_0 and $\mathbf{r}$.

3. Show that the motion equation can be written as

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \omega_0^2\mathbf{r} = 0$$

where the amplitude damping coefficient γ depends on the physical parameters.

4. $1\,\mathrm{Ry} = mc^2\alpha_f^2/2$ being the ionisation energy of the hydrogen atom, give an order of magnitude of the pulsation ω_0 and the wavelength associated to the system. The constant $\alpha_f = 1/137$ is here the fine structure constant. Derive the order of magnitude for the quality factor of the oscillating electron.

Classical polarisability

The electron is set in movement by an incoming oscillating field $E_0\exp(-i\omega t)\mathbf{e}_z$. In the steady state $\mathbf{r} = \mathbf{r}_0\exp(-i\omega t)$ and the oscillating dipole created by the moving electron is defined as $\mathbf{d} = \mathbf{d}_0\exp(-i\omega t)$.

5. Write the equation of motion in presence of the electric field. Deduce the classical polarizability α_c defined as $\mathbf{d}_0 = \epsilon_0\alpha_c E_0\mathbf{e}_z$.

6. From the Larmor formula, write the total average radiated power as a function of the classical polarizability α_c.

Polarisation density for an atomic ensemble

From the Maxwell equation in matter, the electric displacement is defined as

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P},$$

with $\mathbf{P}$ the density of polarisation of the bound charges. For a density $\mathcal{N}$ of independant atoms, the density of polarisation can be rewritter $\mathbf{P} = \epsilon_0\chi_C\mathbf{E}$ with $\chi_C = \mathcal{N}\alpha_C$.

7. Deduce the relative permittivity ϵ_r such that $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$.

The density of energy given by the field into the matter is given by

$$\mathcal{E} = \frac{1}{2}\mathrm{Re}\left(-i\omega\mathbf{P}\cdot\mathbf{E}^*\right).$$

8. Write this density of energy as a function of α_c. Does this model explain the light field amplification effect?

Remark: This classical description of the atom agrees with the quantum mechanical treatment in the limit where the saturation of the transition is negligible, be it due to low beam intensity, or large detuning of the light from atomic resonance. Here the amplitude damping coefficient corresponds to the linewidth of the atomic transition. In the quantum mechanical description, $\gamma = \omega_0^3|\mu|^2/(3c^3\pi\epsilon_0\hbar)$ is the spontaneous population decay rate of the atomic excited state, given in terms in the dipole matrix element $\mu = \langle e|\mu|g\rangle$ between the ground state $|g\rangle$ and the excited state $|e\rangle$.

Exercise 2: Emission, absorption and dispersion by an atom

This exercise is a logical continuation of the previous one, in which we set out to explain, still using a classical model, the principle of emission, absorption and scattering by an atom or group of atoms.

We analyse the interaction of a single atom, described as a pointlike classical dipole oscillator, with a Gaussian TEM_{00} mode of wave number $k = 2\pi/\lambda = \omega/c$ and a waist w that is at least somewhat larger than an optical wavelength λ such that the paraxial approximation for the propagation of Gaussian beams is valid.

The atom polarizability α is now slightly different from the one derived in exercise 1, given by

$$\alpha(\omega) = 6\pi\epsilon_0 c^3\frac{\gamma/\omega_0^2}{\omega_0^2 - \omega^2 - i\left(\omega^3/\omega_0^2\right)\gamma}$$

and obeys the relation

$$|\alpha|^2 = \frac{6\pi\epsilon_0}{k^3}\mathrm{Im}(\alpha).$$

In presence of a driving field $\mathbf{E}_I = 1/2E_I e^{-i\omega t}\mathbf{e} + \mathrm{c.\,c}$, the oscillating dipole emits a radiation field whose amplitude at large distance $R \gg \lambda$ from the atom is given by

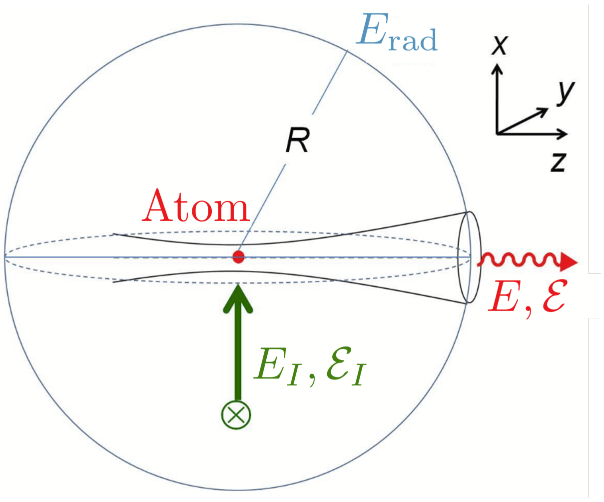
$$E_{\mathrm{rad}}(R,\theta) = \frac{k^2\sin(\theta)}{4\pi\epsilon_0}\frac{e^{ikR}}{R}\alpha E_I$$

where θ is the angle between the polarization $\mathbf{e}$ of the driving field and the direction of observation.

The field radiated into the same mode as the driving field can interfere with the latter, resulting in absorption and emission effects.

Scattering into a free space mode: emission

An incident field E_I is polarized perpendicular to the TEM$_{00}$ mode and drives an atomic dipole oscillator that emits an electromagnetic field E_{rad} at large distance R from the atom. The atom is located on the axis at the waist of the TEM$_{00}$ mode, as shown in the figure below.



The TEM$_{00}$ field at position (ρ, z) can be written for $z \gg z_R$ as

$$E(\rho, z) = \frac{\mathcal{E}}{\sqrt{\epsilon_0 c}} e(\rho, z), e(\rho, z) \simeq \sqrt{\frac{2}{\pi \tilde{w}^2}} \exp\left(-\frac{\rho^2}{\tilde{w}^2} + ikz + ik\frac{\rho^2}{2z} - i\frac{\pi}{2}\right) \text{ and } \tilde{w}(z) = w\sqrt{1 + (z/z_R)^2} \simeq wz/z_R$$

with $\mathcal{E}$ the mode amplitude which can be related to the total power $P = |\mathcal{E}|^2/2$ and to the electric field at waist $\mathcal{E} = E(0,0)\sqrt{\epsilon_0 c A}$ where $A = \pi w^2/2$ is the effective mode area. We can similarly define a mode amplitude for the driving field $\mathcal{E}_I = E_I\sqrt{\epsilon_0 c A}$, even if the driving field is some arbitrary mode.

1. Show that the TEM$_{00}$ mode amplitude arising from the radiated field can be expressed as $\mathcal{E} = i\beta\mathcal{E}_I$ with the dimensionless parameter

$$\beta = \frac{k}{\pi w^2} \frac{\alpha}{\epsilon_0}$$

Hints: The mode amplitude $\mathcal{E}$ can be calculated as the projection $\mathcal{E} = \sqrt{\epsilon_0 c} \int e^*(\rho, z) E_{\text{rad}} 2\pi\rho \mathrm{d}\rho$ in the plane $z \gg z_R$. The spatial dependence of the emitted dipole field E_{rad} over the region occupied by the TEM$_{00}$ can be approximated as $\sin\theta \simeq 1$ and $e^{ikR}/R \simeq e^{ikz + ik\rho^2/(2z)}/z$.

In the following, we will use another dimensionless parameter called the single-atom cooperativity η defined as

$$|\beta|^2 = \eta \times \mathrm{Im}(\beta)$$

2. Give the mathematical value of η.

3. Recover the total scattered power into all directions P_{rad} obtained from Exercice 1. Express it as a function of η and $\mathcal{E}$.

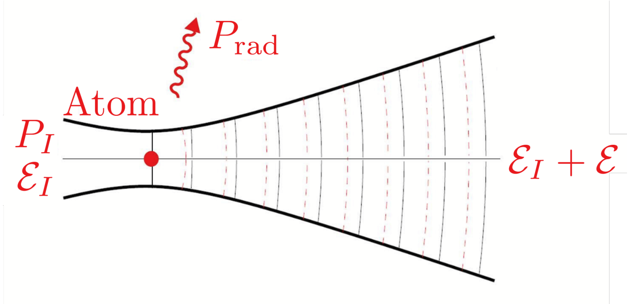
Hint: The total power into all directions P_{rad} can be calculated by integrating over the intensity $I_{\text{rad}} = \epsilon_0 c |E_{\text{rad}}|^2/2$ of the radiated field over the surface of the sphere of radius R.

4. Express the power emitted into both directions of TEM$_{00}$ mode $2P$.

5. Give a physical interpretation of η.

Scattering from a free space mode: absorption and dispersion

We consider the same TEM$_{00}$ mode as in the previous section but now take the light to be incident in that mode with power $P_I = \frac{|\mathcal{E}_I|^2}{2}$, as shown in the figure below.



6. We propose to derive the fractional attenuation $P_{\rm abs}/P_I$ with $P_{\rm abs}$ the power absorbed from the driving field by two different ways :

- a) by using the energy conservation, i.e. the power $P_{\rm rad}$ scattered by the atom must be equal to $P_{\rm abs}$.
- b) by using the idea that the power reduction in the forward direction must be arising from destructive interference between the incident field $\mathcal{E}_I$ and the field $\mathcal{E} = i\beta\mathcal{E}_I$.

In general, the driving field is not only attenuated but also experiences a phase shift in the presence of the atom. This phase shift, corresponding to the atomic index of refraction can be simply understood as arising from the interference of the out of phase component of the forward-scattered field by the atom $\mathcal{E}$ with the incident field in the same mode $\mathcal{E}_I$.

7. Show that this atom-induced phase shift can be written as $\Phi = \text{Re}(\beta)$.

Within the rotating wave approximation (RWA), $\Delta = \omega - \omega_0 \ll \omega_0$, the mode coupling parameter β in terms of the light-atom detuning Δ takes the simple form

$$\beta_{\rm RWA} = \eta\left(\mathcal{L}_d(\Delta) + i\mathcal{L}_a(\Delta)\right)$$

where $\mathcal{L}_d(\Delta) = -2\Delta\gamma/(\gamma^2 + 4\Delta^2)$ and $\mathcal{L}_a(\Delta) = \gamma^2/(\gamma^2 + 4\Delta^2)$ are the Lorentzian dispersive and absorptive lineshapes respectively.

8. Plot both the fractional attenuation and the phase-induced shift in function of Δ/γ Comment the results.

Absorption and dispersion by an atom ensemble

For an ensemble of N atoms located on the mode axis, the total fractional attenuation equals N times the single-atom fractional attenuation. Similarly, the phase shift induced by the ensemble on the light is N times the single-atom phase shift.

9. Write both the fractional attenuation and the phase-induced shift at resonance and at large detuning $\Delta \gg \gamma$ in presence of N atoms. How can we define the collective cooperativity ?

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