Quantum mechanics II, Problems 9- Perturbation Theory III

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Problem 1 : Time-dependent Perturbation Theory - Perturbed Harmonic Oscillator

Let's consider a harmonic oscillator described by

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \tag{1}$$

on which a sudden electric field $\hat{V} = -F\hat{x}$ is suddenly applied.

Recall the eigenfunctions of the harmonic oscillator i.e. $\hat{H}_0$:

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) ,$$

where the functions H_n are Hermite polynomials defined as follows

$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}\left(e^{-z^2}\right) .$$

1. Determine the exact transition probability $P_{\psi_0^{(0)}\to\psi_n}$ between an initial state and an excited state following this perturbation.
   (Hint : you should be able to rewrite the full Hamiltonian into a well known form i.e. for which solutions to Schrödinger equations are known)
2. Under what condition does $\hat{V}$ correspond to a weak perturbation ? In this limit, determine $P_{\psi_0^{(0)}\to\psi_1}$.
3. Find the expression of $P_{\psi_0^{(0)}\to\psi_1}$ using first-order time-dependent perturbation theory and compare it to the previous result.

Problem 2 : Two degenerate states

Let's consider a system described by a Hamiltonian $\hat{H}_0$ with 2 eigenstates $\psi_1^{(0)}$ and $\psi_2^{(0)}$ having the same energy E. We perturb the system as follows : $\hat{H} = \hat{H}_0 + \hat{V}$.

1. At first order according to perturbation theory, the eigenstates become :

$$\psi^{(0)} = c_1^{(0)}\psi_1^{(0)} + c_2^{(0)}\psi_2^{(0)} \quad \text{and} \quad \psi'^{(0)} = c_1'^{(0)}\psi_1^{(0)} + c_2'^{(0)}\psi_2^{(0)} \tag{2}$$

   What are the values of coefficients $c_{1,2}^{(0)}$ and $c_{1,2}'^{(0)}$?
2. If the initial state is $\psi_1^{(0)}$, what is the probability of finding the system in state $\psi_2^{(0)}$ at time t ? Show that for short times we obtain :

$$P_{1\to 2}(t) = \frac{t^2}{\hbar^2}|V_{21}|^2. \tag{3}$$

3. Retrieve the result (3) through time-dependent perturbation theory.