Quantum Physics II
Lectures Notes on Group Theory

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0. Symmetry in Quantum Mecanics

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A Symmetry is a transition that leaves the physical property invariant. A large part of theoretical physics is about the consequences of symmetry. P.W. Anderson, Physics Nobel prize in 1972, once said "It is only slightly overstating the case to say that physics is the study of symmetry."

0.1 Symmetry in quantum mechanics

Let us consider some set of transformations that, somehow, do not change the physical properties of a system (for instance, a rotation) This means that we want to apply an operator to a ket describing the physical system. What properties do we want these transition to have?

- If we transform the system as $|\psi'\rangle = \hat{D}|\psi\rangle$, we want the inverse transformation to exist, so that $\hat{D}^{-1}|\psi'\rangle = |\psi\rangle$.
- We want to be able to apply these transformations in the set, that is we want to be able to do a transformation, and then another one, as : $|\psi'\rangle = \hat{D}_\alpha \hat{D}_\beta |\psi\rangle = \hat{D}_\gamma |\psi\rangle$. In particular, we want $D_\alpha^-1 D_\alpha = 1$, so we also need a neutral element.
- Given we can apply these transformation one by one, we also require an associative law $(D_\alpha D_\beta)D_\gamma = D_\alpha(D_\beta D_\gamma)$.

This, already, tells us that we want to represent a set of transformations as a group, in the mathematical sense! This is one of the reason (and actually the main one in this course) why we are going to study group theory.

0.2 Invariance, commutation, Wigner and Noether theorem

A fundamental point is that "invariance" with a group transformation shows that the action of the group on the physical reality is very constraint. For instance we see that if a system is invariant under a transformation, then the new ket $|\psi'\rangle = \hat{D}|\psi\rangle$ must satisfies $|\langle\phi'|\psi'\rangle|^2 = |\langle\phi|\psi\rangle|^2$. This already, is a strong limitation on what $\hat{D}$ could be! Indeed we see that if we want

$$\langle \phi' | \psi' \rangle = \langle \phi | \psi \rangle$$

we need $\hat{D}^\dagger \hat{D} = \mathbb{I}$: in this case, we sat that D is an unitary operator [1] so that $\hat{D}^\dagger = \hat{D}^{-1}$.

We shall see later on that indeed Wigner's theorem tells us indeed that symmetry can only be represented by such a unitary operator. This means that we should be able to write a symmetry operator as

$$\hat{D} = e^{i\hat{T}}$$

[1] Note that this is not the only possibility, since strictly speaking quantum mechanics require the SQUARE to be equal, and indeed, this leaves another possibility, called anti-unitary operator. We shall see that this will be used (only) for one type of symmetry, the so-called "time reversal" symmetry.

with T Hermitian (that is, with $\hat{T}^\dagger = \hat{T}$. Remember that the exponential of an operator is defined as

$$e^A = \sum_{k\in\mathbb{N}} \frac{1}{k!} A^k$$

. This is one of the most important aspect of "representation theory", that is the representation of symmetry transformation as linear operator, which will keep us busy in the next weeks.

Let us see the consequences: if we measure an observable, the transformation must have no effect, and this means that

$$\langle \hat{O}\rangle' = \langle \psi' | \hat{O} | \psi' \rangle = \langle \psi | \hat{O} | \psi \rangle \langle = \hat{O} \rangle \tag{1}$$

This is only possible if

$$\langle \psi' | \hat{O} | \psi' \rangle = \langle \psi | \hat{D}^{-1} \hat{O} \hat{D} | \psi \rangle = \langle \psi | \hat{O} | \psi \rangle \tag{2}$$

so that

$$\hat{D}^{-1} \hat{O} \hat{D} = \hat{O} \tag{3}$$

so that

$$\left[\hat{D}, O\right] = 0 \tag{4}$$

A transformation that left a quantity invariant must therefore COMMUTE with it! In particular, a transformation that does not effect the Hamiltonian must satisfy

$$\left[\hat{D}, H\right] = 0 \tag{5}$$

We see that symmetries are therefore associated with commutation properties! this will be a central point in the following.

This has an immediate consequence. Remember that we can write the time evolution of an operator (in the Heisenberg picture) as follow:

$$\hat{O}(t) = e^{i\frac{\mathscr{H}}{\hbar}t} \hat{O} e^{-i\frac{\mathscr{H}}{\hbar}t} \tag{6}$$

This directly follows from Schrodinger equation [2]. But if $\mathscr{H}$ commute with O, then its exponential also commutes with O. So that we find

$$\hat{O}(t) = e^{i\frac{\mathscr{H}}{\hbar}t} \hat{O} e^{-i\frac{\mathscr{H}}{\hbar}t} = \hat{O} \tag{7}$$

This means that if a transformation commute with the Hamiltonian, which is our definition of symmetry, then we have a conservation law. This can also be deduce by the relation

$$\frac{d}{dt}\hat{O}(t) = \frac{1}{i\hbar}\left[\hat{O}, \mathscr{H}\right] \tag{8}$$

What does this means for a symmetry $\mathscr{D}$? Well it means that operator that represents symemtry are time invariant. But since we know that $\hat{D} = e^{i\hat{T}}$ this prove that there is a non-trivial operator $\hat{T}$ that must be also time-invariant, and that $\langle \hat{T} \rangle$ is conserved: the presence of a symmetry implies a law of conservation! This is an instance of a *Noether theorem*, one of the most important theorems in mathematical physics! In fact, one can show that time invariance implies the conservation of energy, translation invariance implies the conservation of impulsion/momentum, and rotation invariance implies the conservation of angular momentum!

[2]From $i\hbar\frac{d}{dt}|\psi(t)\rangle = \mathscr{H}|\psi(t)\rangle$, we can write $|\psi(t)\rangle = e^{-i\frac{\mathscr{H}}{\hbar}t}|\psi(0)\rangle$. Going from Schrodinger to Heisenberg picture leads to the time evolution.

0.3 Symmetry and selection rules

A fundamental application of symmetry are selection rules: It is possible to prove that, in a certain basis, many elements of the Hamiltonian matrix are simply zero! This is usually very helpful: if these are null, then we do not need to compute (or to measure) them.

A simple example is given by the particle in a box. In this case the Hamiltonian reads

$$\left(-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right) \tag{9}$$

Assume the potential is even, Then we have a symmetry when transforming x into $-x$! This is the parity transformation.

As we shall see this transformation has a deep consequence. Assume we write the solution on a basis of even and odd functions (Notice that even functions are invariant under parity, while odd functions transform into another odd function, but two such transformations bring it back to the original) such as cosinus and sinus.

A fundamental mathematical result (Schur lemma) tells us that, in this basis (let us say the upper part of the basis is even, and the lowest part is odd) the Hamiltonian can be partitioned as follows

$$\mathscr{H}=\begin{bmatrix}H_1 & 0\\ 0 & H_2\end{bmatrix} \tag{10}$$

This is great, we already do not need to compute half of the coefficients! Additionally, since time evolution is given by applying the Hamiltonian, we see that if we initialize at time zero in a even function, it will stay this way at all time. If, on the other hand, we initialize in an off fucntion, if will also stay even at all times. This will be true for any problem respecting the Parity symmetry.

Finding such meaningful division, or partition, into different sector is the main object of the representation theory of group in quantum mechanics.

0.4 Symmetry and degeneracy

A final important application of symmetry is that they imply a (possible) degeneracy. Say that we found an eigenvector of the hamiltonian. Then we have:

$$\mathscr{H}\left|\psi\right\rangle = E\left|\psi\right\rangle \tag{11}$$

But if we have a symmetry, then it exists $\hat{D}$ such that $[\hat{D},\mathscr{H}]=0$ then

$$\hat{D}\mathscr{H}\left|\psi\right\rangle = \hat{D}E\left|\psi\right\rangle \tag{12}$$

$$\mathscr{H}\hat{D}\left|\psi\right\rangle = E\hat{D}\left|\psi\right\rangle \tag{13}$$

$$\mathscr{H}\left(\hat{D}\left|\psi\right\rangle\right) = E\left(\hat{D}\left|\psi\right\rangle\right) \tag{14}$$

So $\hat{D}\left|\psi\right\rangle$, too, is an eigenvector with the same energy. If it so happens (it does not have to be, but it may) that $\hat{D}\left|\psi\right\rangle$ is independent of $\left|\psi\right\rangle$, then you have another eigenvector with the same energy level, i.e. a degeneracy.

Physicists usually think the other way around: if there is a degeneracy, they look for the symmetry that cause it, and usually, there is one (a degeneracy without a symmetry is called an accidental degeneracy, but this is not the typical situation).

Let us see an example with the Hydrogen atom. The Schordinger equation uses the Hamiltonian:

$$\mathscr{H}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}-\frac{\hbar^2}{mr}\frac{\partial}{\partial r}+\frac{\hat{L}}{2mr^2}+V(r) \tag{15}$$

Note there is a radial symmetry, it only depends on "r", not "x", "y" or "z". So there is a rotation symmetry in this problem. Now, let us look into the solution. We know the energy levels reads

$$E_n = -\frac{\hbar^2}{2n^2} \tag{16}$$

For $n = 1$, there is a single solution, the "1s" state. However, it has a perfectly symmetric eigenfunction, so indeed rotation cannot change it.

For $n = 2$, there are 3 solutions called $2px, 2py, 3pz$. Indeed they are simply rotations of one another! Degeneracy is just given by application of the symmetry. There is, however, the "2s" state which can not be obtained by rotation of the $2p$ states.

Is this an "Accidental" degeneracy then? No! This is the case, there is a symmetry called SO(4) behind this, as we shall see in the exercise session (Serie 6).

1. Element of Group Theory

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1.1 Formal definition

Definition 1.1.1 A group is a set equipped with an operation that combines any two elements to form a third element while being associative as well as having an identity element and inverse elements.

Formally, one can write a set G equipped with the operation "$\cdot$" is a if one has:

- Associativity: $\forall a,b,c \in G$, one has $(a\cdot b)\cdot c = a\cdot(b\cdot c)$.
- An Identity element: There exists an element $e \in G$ such that $e\cdot a = a \forall a \in G$. Such an element is unique (see below) and is called the identity of the group.
- Inverse element: $\forall a \in G$, it exists $b \in G$ such that $b\cdot a = e$. We then say that $b = a^{-1}$. For each e the element a^{-1} is unique (see below) and is called the inverse of a.

The group axioms imply that the identity element is unique: If e and f are both identity elements of a group, then $e = e\cdot f = f$.

Similarly, they also imply that the inverse of each element is unique: If a group element a has both b and c as inverses, then

$$b \quad = \quad b\cdot e = b\cdot(a\cdot c) = (b\cdot a)\cdot c = c \tag{1.1}$$

Another useful remark is that the inverse of ab is $b^{-1}a^{-1}$ since, indeed $b^{-1}a^{-1}ab = e$.

Here are some examples of groups:

- $(\mathbb{R},+)$: the set of real equipped with the addition, is a group. 0 is the neural element, and each real a has an inverse $-a$.
- $(\mathbb{R}^*,*)$: the set of reals *minus 0* equipped with multiplication, is a group. 1 is the neural element, and each real a has an inverse $a^{-1} = 1/a$ (this is why zero has to be excluded).
- $(\{+1,1\},*)$ is a group. The neural element is 1.
- The set of all invertible matrices in dimension d equipped with matrix multiplication is a group.

Definition 1.1.2 — Abelian and non-Abelian groups. : If $a\cdot b = b\cdot a \forall a,b \in G$, the group G is said to be abelian. Otherwise it is called a non-Abelian group. These groups are also called commutative and non-commutative.

For instance, while $(\mathbb{R},+)$ is abelian, the group of all intertible matrices in dimension larger than 1 is non-abelian.

1.2 Finite Groups and Lie groups

Definition 1.2.1 — Finite group. A group that contains a finite number of element is called a finite group. The number of element is called the *order* of the group.

Finite group will be our primal concern in this course. There are many interesting finite groups, among which the simplest is the cyclic group Z_n. It is a group all of whose elements are powers of a particular element a where $a^n = a^0 = e$. For instance $Z_2 = [1,-1]$. Here $e = 1$, $a = -1$ and $a^2 = 1 = e$. The nth roots of unity in the complex plane $a_j = e^{i2\pi\frac{j}{n}}$ equipped with multiplication forms a groups Z_n, since $a_j = a^j$ and $a^n = 1$. Another example is the additive group of the integers modulo n. In fact these two groups –as we shall see– are the same one, and are said to be "isomorphic".

Of course, what will be very important for us are symmetry groups! These are given by a set of transformation that form a group. The simplest one is the Parity group that contains the "transformation in the mirror" that turns x into $-x$. Let us define the operator $\hat{P}$ such that $\hat{P}f(x) = \hat{P}f(-x)$. Given $\hat{P}\hat{P} = 1$, we see that the set of transformation $\{1, \hat{P}\}$ form a group. We shall see this group is isomorphic to Z_2.

Another example of a symmetry group is the set of $2D$ rotations that leaves a square invariant (this is isomorphic to Z_4).

In what follows, we shall consider very often, for concreteness the group of symmetry of the Ammonia molecule, NH3. This is called the $C3V$ group. There are 5 possible transfomation that leaeves the moleculre invariance:

a) the identity E which leaves all coordinates unchanged.

b) the proper rotation Cn by an angle of 2π/n in the positive trigonometric sense (i. e. counter-clockwise). There are two of them. c) Symmetry in the plane, there are three of them.

Definition 1.2.2 — Lie group. Informally, a Lie group is a continuous group that depends *analytically* on some continuous parameters λ.

A classic example is the group of orthogonal $2d$ matrices, where all element can be written as

$$M(\phi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}. \tag{1.2}$$

The group corresponding to orthogonal matrices in dimension n is denoted $O(n)$. The one corresponding to orthogonal matrices with determinant one is $SO(n)$ (special orthogonal group). The unitary group of degree b (The group of $n \times n$ unitary matrices) is denoted $U(n)$, and again $SU(n)$ corresponds to those with determinant 1.

A lie is a group that is also a differentiable manifold. In the former example, the manifold is the circle parameterized by $\phi \in [0; 2\pi]$. There is a deep relation between topology and the Lie Group.

Note that not all infinite groups are Lie groups! The set of all rational numbers equipped with addition is infinite (but countable), but it is not a Lie group.

1.3 Sub-groups

Definition 1.3.1 — sub-group. A subset H of the group G is a subgroup of G if and only if it is nonempty and closed under products and inverses.

The closure conditions mean the following: Whenever a and b are in H, then $a \cdot b$ and a^{-1} are also in H.These two conditions can be combined into one equivalent condition: whenever a and b are in H, then $a \cdot b^{-1}$ is also in H.

R The identity of a subgroup is the identity of the group: if G is a group with identity e_G, and H is a subgroup of G with identity e_H, then $e_H = e_G$.

R We call a subgroup of G which is neither the identity nor G itself a *proper* subgroup.

A fundamental result in the theory of finite groups is Lagrange theorem:

Theorem 1.3.1 — Lagrange. Let G be a finite group and H a subgroup of G, then the order of H divides the order of G.

We prove this theorem in sec.1.7.

It is easy to see[1] that this implies in particular that if the order n if a finite group G is a prime, then there are only one possible group: the cyclic one Z_n.

1.4 Cayley table for finite groups

The study of finite group has been a fundamental topic in mathematics over the last two centuries. Many results are known, and there is an entire classification of these groups, a topic well beyond the topic of this course. A unique way to describe a group is to write its Cayley table.

Named after the 19th century British mathematician Arthur Cayley, a Cayley table describes the structure of a finite group by arranging all the possible products of all the group's elements in a square table reminiscent of an addition or multiplication table. Many properties of a group – such as whether or not it is abelian, which elements are inverses of which elements, and the size and contents of the group's center – can be discovered from its Cayley table:

$$\begin{array}{c|ccccc} * & 1 & a & a^2 & a^3 & a^4 \\ \hline 1 & 1 & a & a^2 & a^3 & a^4 \\ a & a & a^2 & a^3 & a^4 & a^5 \\ a^2 & a^2 & a^3 & a^4 & a^5 & a^6 \\ a^3 & a^3 & a^4 & a^5 & a^6 & a^7 \\ a^4 & a^4 & a^5 & a^6 & a^7 & a^8 \end{array} \tag{1.3}$$

Here are few example of all possible low-order groups:

- **Order 1** :There is only one such group, the trivial one:

$$\begin{array}{c|c} * & e \\ \hline e & e \end{array} \tag{1.4}$$

- **Order 2** : There is only one such group, the cyclic one Z_2:

$$\begin{array}{c|cc} * & e & a \\ \hline e & e & a \\ a & a & e \end{array} \tag{1.5}$$

- **Order 3** : By Lagrange, this too is unique.

$$\begin{array}{c|ccc} * & e & a & b \\ \hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array} \tag{1.6}$$

- **Order 4** : This is the smallest order that is not unique, as there are two order 4 groups: the cyclic group Z_4, as well as another one (who has a subgroup of order 2). These are their Cayley table:

[1] One way to create a subgroup is to pick one element of the group and to construct a cyclic group from it. This will either coincide with the group itself (if it is a cyclic group) or it will create a subgroup. If a finite group G of order n has no subgroup, then this implies that it must be the cyclic group Z_n.

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(1.7)

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

(1.8)

R Notice how this group has some subgroups of order 2: ($\{e,a\}$, $\{e,b\}$,$\{e,c\}$).

- **Order 5** : By Lagrange, this too is unique, and is a cyclic group.

$*$	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	d	d	e	a
c	c	e	e	a	b
d	d	e	a	b	c

(1.9)

- **Order 6** : One can write again the Z_6 cyclic group, but there is a second group of order 6 which is very interesting, as it is the lowest order one that is non-abelian:

$*$	e	a	a^2	b	c	d
e	e	a	a^2	b	c	d
a	a	a^2	e	c	d	b
a^2	a^2	e	a	d	b	c
b	b	d	c	e	a^2	a
c	c	b	d	a	e	a^2
d	d	c	b	a^2	a	e

(1.10)

R Notice how this group has subgroups of order 2: ($\{e,b\}$, $\{e,c\}$,$\{e,d\}$ and one subgroup of order 2: ($\{e,a,a^2\}$,

There are few things worth noticing in the Cayley table. First obviously, the first line is the repetition of the "zero"th line, as well as the first column. This is a simple property of the neutral element. More interestingly, we see that each line and each row always contain all elements, and only once. In other words, each line is a permutation of the other lines. This is a consequence of the following theorem:

Theorem 1.4.1 — Reordering theorem. Let G be a group and m one of its elements. Then the application $G \to mG$ and $G \to Gm$ are bijective. .The ensembles mG and Gm are thus re-order of G.

Proof. The map $x \to mx$ is surjective all elements have an antecedent). Indeed for any $y \in G$, $m^{-1}y \in G$ (group property) and $m(m^{-1}y) = y$. The map $x \to mx$ is also injective (it maps distinct elements to distinct elements). For any $x \neq x'$, $mx \in G$ is different from mx'. Indeed, if $mx = mx'$ then $m^{-1}(mx) = m^{-1}mx'$ and $x = x'$. The proofs works in a similar way for the map $x \to xm$. ■

1.5 Group Homomorphism and isomorphism

A group morphism, or group homomorphism, is a mapping between two groups which respects the group structure:

Definition 1.5.1 — Group morphism or homomorphism. A morphism from a group $(G,*)$ in the group $(G',\star)$ is an application $f : G \to G'$ such that $\forall x,y \in G \quad f(x*y) = f(x)\star f(y)$.

It implies in particular that $f(e) = e'$, (where e and e' denote the respective neutrals of G and G') as well as $f(x^{-1}) = f(x)^{-1}$. For instance, it is always possible to create a morphism of any finite group to the trivial group by mapping all the elements to e'.

Homomorphism from $f : G \to G'$ can be surjective, this is the case when $G' = \mathrm{Im}(G)$. This will be often the case in this lecture. They can, too, be bijective. In this case, we call them isomorphism.

Definition 1.5.2 — Group isomorphism. A group isomorphism is a function between two groups that sets up a one-to-one correspondence between the elements of the groups in a way that respects the given group operations.

If there exists an isomorphism between two groups, then the groups are called isomorphic. From the standpoint of group theory, isomorphic groups have the same properties and need not be distinguished. Let us give some examples:

- The function $f(x) = 1$ allows to show that that the group $\{1,*\}$ is always an homomorphism of any group. This is called the trivial homomorphism. As trivial as it is, we shall see that it is immensely useful!
- A less trivial example is that the group Z_2 is homomorphic to $Z = \{\dots,-3,-2,-1,0,1,2,\dots\}$ equipped with addition using $f(x) = 1$ for even numbers and $f(x) = -1$ for odd numbers.
- An example of isomorphism of Z_n is given by using $[1,n]$ with addition modulo n, or equivalently by using $e^{2pik/n}$.

Theorem 1.5.1 — Image of inverses and neutral element. If G is homomorphic to H by $f : H \to G$, then $e' = f(e)$ and $f(u^{-1}) = (f(u))^{-1}$

Proof. Clearly, $f(e)f(u) = f(eu) = f(u)$ so the first property is proven. For the second, we write $f(u)f(u^{-1}) = f(uu^{-1}) = f(e) = e'$ so that the converse of $f(u)$ is $f(u)^{-1}$ ■

1.6 Equivalence and Conjugacy classes

Definition 1.6.1 — Equivalence relation. A binary relation $\sim$ on a set X is said to be an equivalence relation, if and only if it is reflexive, symmetric and transitive. That is, for all $a,b,c \in X$, we have:

- $a \sim a$ (*Reflexivity*)
- $a \sim b \Rightarrow b \sim a$ (*Symmetry*)
- $(a \sim b),(b \sim c) \Rightarrow a \sim c$ (Transitivity)

Equivalence relation allows to divide a set into disjoints set called "Equivalence classes".

1.6.1 Conjugacy class

Let G be a group, we define the following equivalence relation: x and y are equivalent if it exists $u \in G$ such that $u^{-1}xu = y$. When then say they belong to the same Conjugacy class.

This is indeed an equivalence relation:

- Using $u = x^{-1}$ we have $x^{-1}xx = x$ so that $x \sim x$ (Reflexively).
- If it exists u such that $u^{-1}xu = y$ then $x = uyu^{-1} = (u^{-1})^{-1}yu^{-1}$ so that using $v = u^{-1} \in G$, $\exists v \in G \mid x = v^{-1}yv$. (Symmetry).

- if $a \sim b$ and $b \sim c$, then it exists $u, v \in G$ sucth that $u^{-1}au = b$ and, by symmetry, $v^{-1}cv = b$ so that $u^{-1}au = v^{-1}cv$ and $vu^{-1}auv^{-1} = c$. Using $w = vu^{-1}$ and $w^{-1} = uv^{-1}$ yields $w^{-1}aw = b$. (Transitivity).

By this theorem, the conjugation relation divides the elements of group G into distinct classes which are called conjugate classes or simply classes.

Let us consider for example the order 4 groups:

$$G = \begin{array}{c|cccc} * & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array} \tag{1.11}$$

In this case, can check that the equivalence class divide the group into four classes, each containing one member. In fact, this is true for each abelian group (and the converse is true). This is a trivial consequence of commutation!

Another interesting example is given y the C_3V group. Here we have three equivalent classes: $\{e\}$,$\{C_+, C_-\}$, and the three mirrors $\{\sigma_1, \sigma_2, \sigma_3\}$. Note that e is always a "isolated" class in itself. Indeed, if $x = u^{-1}eu$ then $x = e$.

The following fact will be useful:

Theorem 1.6.1 — "Composition" of conjugacy classes. Let G be a group, and C_x and C_y two of its conjugacy classes. Then we have

$$C_\nu \cdot C_\mu = \sum_\lambda n_{\mu\nu\lambda} C_\lambda \tag{1.12}$$

with $n_{\mu\nu\lambda}$ integer. Here the multiplication $C_\nu \cdot C_\mu$ is defined as the entire set $[xy]$ for all $x \in C_\nu$ and $y \in C_\mu$. Additionally, $n_{\nu\mu\lambda} = n_{\mu\nu\lambda}$ and $n_{1\nu\lambda} = n_{\nu 1\lambda} = \delta_{\nu,\lambda}$.

We prove this theorem in section 1.8.

1.6.2 The group C3v

A particularly interesting group is the symmetry group of the Ammonia molecule NH3. In addition to the identity, it has 3 mirrors and 2 rotations. Since it is a finite group of order 6, it must be isomorphic to one of the groups we already wrote. Indeed its Caley table reads:

$$\begin{array}{c|cccccc} * & e & c_3^1 & c_3^2 & \sigma & \sigma' & \sigma'' \\ \hline e & e & c_3^1 & c_3^2 & \sigma & \sigma' & \sigma'' \\ c_3^1 & c_3^1 & c_3^2 & e & \sigma' & \sigma'' & \sigma \\ c_3^2 & c_3^2 & e & c_3^1 & \sigma'' & \sigma & \sigma' \\ \sigma & \sigma & \sigma'' & \sigma' & e & c_3^2 & c_3^1 \\ \sigma' & \sigma' & \sigma & \sigma'' & c_3^1 & e & c_3^2 \\ \sigma'' & \sigma'' & \sigma' & \sigma & c_3^2 & c_3^1 & e \end{array} \tag{1.13}$$

When we construct the conjugacy classes, we find "e", then the rotations c_3^1, c_3^2, and finally the class σ, σ, σ''. Interestingly these classes seem to mean something! They group the rotation and the mirrors together!

1.7 Proof of Lagrange Theorem, Right and Left cosets

Let G be a group and H one of its proper subgroups. We can define an equivalence relation —different from the last one— between the elements of G as follows: if $x, y \in G$ and $x^{-1}y \in H$ then

x and y are equivalent and we write $x \sim y$.

This is indeed an equivalence relation:

- $a^{-1}a = e \forall a \in G$, and $e \in H$, so that $a \sim a$.
- if $a \sim b$ then $a^{-1}b \in H$. The inverse of $a^{-1}b$ is $b^{-1}a$ and since H is a group, $b^{-1}a \in H$, so that $b \sim a$.
- if $a \sim b$ and $b \sim c$, then $a^{-1}b$ and $b^{-1}c$ are both in H, thus so is the their product $a^{-1}bb^{-1}c = a^{-1}c$.

This equivalence relation therefore makes it possible to divide the elements of G into disjoint classes. If $x^{-1}y \in H$, then y is equal to an element of H multiplied on the left by x. We indicate the set thus constructed by the symbol

$$C_x = xH \tag{1.14}$$

which the call the *left co-set associated to x*.

The map $H \to xH$ is one-to-one (bijective). Indeed, each element $z \in xH$ is the image of $x^{-1}z \in H$ so that the map is surjective. But the map is also injective since for $y, y' \in H$, we have $xy = xy' \Rightarrow y = y'$.

We could also define a second equivalence relation $x \sim y$ if $yx^{-1} \in H$ and wihch this case, we can define the concept of right co-set Hx in the same way as before.

These concepts are very useful, and allows in particular to prove Lagrange Theorem:

Proof. Consider the co-sets on the left of H. They are all disjoint or identical (since they are equivalence classes). If there are n distinct left co-sets, their union is G. So, if we denote by g and h the orders of G and H respectively, then $g = nh$ and the theorem is proved. ■

Let us give an example for the following order 4 group

$$G = \begin{array}{c|cccc} * & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array} \tag{1.15}$$

that has the subgroup $H = \{e, a\}$:

$$H = \begin{array}{c|cc} * & e & a \\ \hline e & e & a \\ a & a & e \end{array} \tag{1.16}$$

We can now construct the left co-sets:

$$C_e = eH = \{e, a\} \tag{1.17}$$

$$C_a = aH = \{e, a\} = C_e \tag{1.18}$$

$$C_b = bH = \{b, c\} \tag{1.19}$$

$$C_c = cH = \{c, g\} = C_b \tag{1.20}$$

And we see indeed that we have *two* left co-set of order 2.

1.8 Proof composition of Conjugacy classes

First, let us prove a variant of the reordering theorem:

Theorem 1.8.1 — Reordering theorem within conjugacy classes. Let G be a group, m one of its elements, and C one of the conjugate classes. Then the application $C \to m^{-1}Cm$ is bijective into itself: The ensemble $m^{-1}Cm$ is thus a re-ordering of C.

Proof. First notice that this is a map into itself since for any $y \in C$, $m^{-1}ym \in C$ (conjugacy class property). Second, the map is surjective. Indeed, for any $y \in C$, it exists $x = mym^{-1} \in G$ such that $y = m^{-1}xm$. By definition, x is thus also in C and therefore for all $y \in C$ there is an antecedent in C. Third, the map $x \to mx$ is injective (it maps distinct elements to distinct elements). For any x, x', we have $m^{-1}xm = m^{-1}x'm$ implies that $mm^{-1}xmm^{-1} = mm^{-1}x'mm^{-1}$ so that $x = x'$. ■

We shall soon prove that $n_{\nu\mu\lambda}$ is indeed an integer. But first, let us note indeed that $n_{\nu\mu\lambda} = n_{\mu\nu\lambda}$, because the two sets $C_\nu \cdot C_\mu$ and $C_\mu \cdot C_\nu$ are identical. Indeed

$$C_\nu \cdot C_\mu = [uv] = \left[uv(u^{-1}u)\right] = \left[u(vu^{-1}u)\right] = \left[uvu^{-1}u\right] = \left[(uvu^{-1})u\right] = C_\mu \cdot C_\nu$$

since u represents all the element of C_ν, and since, from the previous theorem, (uvu^{-1}) represent a re-ordering of the all the element of C_μ as v changes. Additionally, we also see that, denoting the class that contains e are C_1, that $C_1 \cdot C_\nu = C_\nu$ so that $n_{1\nu\lambda} = n_{\nu 1\lambda} = \delta_{\nu,\lambda}$.

Let us now prove that $n_{\nu\mu\lambda}$ is an integer. First we prove the following lemma:

Lemma 1.8.2 A necessary and sufficient condition for a set $[R]$ to be composed uniquely of a set of entire classes of a group G is that

$$\forall u \in G u^{-1}[R]u = [R]$$

Proof. The condition is necessary because, if indeed $[R]$ is composed of entire sets, then in each of these sets S, $u^{-1}[S]u$ is itself the set S by the reordering theorem.
To see that the condition is sufficient, let us proceed by contradiction and write

$$[R] = [R'] + [R'']$$

where $[R']$ is the largest subset of $[R]$ made of entire classes, and the reminder $[R'']$ thus must contain elements that are not an entire class. Since $[R']$ satisfy $u^{-1}[R']u = [R']$ then

$$u^{-1}[R'']u = [R''] \,.$$

e cannot be in $[R'']$ since it is, itself, a class. Let us suppose $[R'']$ is not empty, and $x \in [R'']$. Then it must exists $y \in G$, conjugated to x, which is *not* in $[R'']$. Since y is conjugated to x we have $u^{-1}xu = y$ for some $u \in G$. But then since $u^{-1}[R'']u = [R'']$ for all u, y must be in $[R'']$. We have thus reach a contradiction, and $[R'']$ is empty. ■

Now we can proceed. Let H be a finite group of order h and conjugacy classes $C_1 = \{e\}$, $C_2, \dots, C_\mu, \dots, C_{N_C}$ its classes. We shall denote by n_μ the number of elements in the class C_μ and by N_C the total number of classes. We have, of course

$$\sum_{\mu=1}^{N_c} n_\mu = h \tag{1.21}$$

Let C_μ and C_ν be two classes of H, and consider the product

$$C_\mu \cdot C_\nu = [uv] \tag{1.22}$$

where u and v are elements of C_μ and C_ν. Then for each $x \in H$, we have

$$x^{-1}C_\mu \cdot C_\nu x = [x^{-1}uvx] = [x^{-1}u(xx^{-1})vx] = [(x^{-1}ux)(x^{-1}vx)] \tag{1.23}$$

Using the theorem of rearrangement, we see that $[(x^{-1}ux)(x^{-1}vx)]$ is just a reordering of $[uv]$ so that

$$x^{-1}C_\mu \cdot C_\nu x = C_\mu \cdot C_\nu \tag{1.24}$$

Applying lemma 1.8.2 then prove theorem 1.6.1.

2. Group Representation

2.1 Definition and example

Definition 2.1.1 — Group representation. A group representation is a representation of all elements of the group as linear operator over a vector space V, thus more specifically as *matrices*. In mathematical terms, this means that we call representation of the group G a morphism of G in the linear group $GL(V)$.

We shall thus consider a map $D : G \to GL(V)$ that maps every element g of the group G to a matrix $D(g)$ that acts on a vector space V (that will be most of time $\mathbb{R}^d$ or $\mathbb{C}^d$, where d is the dimension of the representation). Note that we should be careful with notation here. $D : G \to GL(V)$ is a map (more precisely a morphism), but $D(g)$ is a matrix, so that $D(g) : V \to V, \forall g \in G$.

Let us give few examples:

- All groups admit a trivial representation (or the Identity representation): $\forall g \in G, D(g) = 1$.
- Consider the group of orthogonal matrices in dimension $d = 3$, called $O(3)$. This is the set of all 3×3 matrices R such that $RR^T = \mathbb{I}$. The morphism $D(g) = \det R = \pm 1$ is a representation of G on the vector space $\mathbb{R}$ (indeed $\det(AB) = \det(A)\det(B)$).
- Thinking of $O(3)$ more fundamentally, one can say it is the group of all transformation that leaves a sphere invariant: these are all rotation and "mirror" transformation. The set of all 3×3 orthogonal matrices is a representation of $O(3)$.
- Consider the parity group $Z_2 = \{e, p\}$ that has the following table:

	e	p
e	e	p
p	p	e

It admits two representations on $\mathbb{R}$: 1) the trivial representation $D(g) = 1$ for $g = e, p$, as well as 2) the representation $D(e) = 1, D(p) = -1$.

Definition 2.1.2 — Regular representation. For a finite group of order h, one can construct the so-called regular representation using $h \times h$ matrices as follows. First start from the following reordered Cayley table (here for $h = 3$):

$$C = \begin{array}{c|ccc} * & e & a^{-1} & b^{-1} \\ \hline e & e & a^{-1} & b^{-1} \\ a & a & e & ab^{-1} \\ b & b & ba^{-1} & e \end{array} \tag{2.1}$$

Now the representation can be done using the following matrices for $g \in G$: We use a matrix which is zero everywhere except for the position that corresponds to the group in the Cayley table:

$$(D_r(g))_{ij} = \delta_{g,C_{ij}} \tag{2.2}$$

With this definition, e is represented by the identity matrix $D_r(e) = \mathbf{1}_d$. It is easy to check that these matrices indeed follow the group algebra.

2.2 Equivalent, Reducible and Irreducible representation

Consider a group G and a representation $D(g)\forall g \in G$. We define now $D'(g) = SD(g)S^{-1}$ where S can be any invertible matrix. This is a similarity transformation[1]. It is easy to see that similarity transformations of representations are still representations.

Theorem 2.2.1 — D' is a representation of G. This is trivially checked: if $D(gh) = D(g)D(h)$ then $D'(gh) = SD(g)D(h)S^{-1} = SD(g)S^{-1}SD(h)S^{-1} = D'(g)D'(h)$.

Definition 2.2.1 — Equivalent representation. Two representations D and D' are equivalent if they are related by a similarity transformation $D'(g) = SD(g)S^{-1}$.

Roughly speaking, representation are equivalent if we can transform one to the other by a linear invertible transformation. If what follow, we shall be mainly concern by unitary represention and transformation. In this case $SS^* = 1$ and $S^* = S^{-1}$. This means that we shall consider two representations as equivalent if they simply correspond to a change of basis: $D'(g) = SD(g)S^T$.

We thus shall consider, in our quantum mechanics lecture, two representations as equivalent if they correspond to a change of basis.

Definition 2.2.2 — Reducible representation. A representation $D(g)$ of a group G is reducible if there exists an invariant subspace. That is, if there exists a non-trivial subspace $W \in V$ such that $\forall |w\rangle \in W$, we have $D(g)|w\rangle \in W$, for any element $g \in G$.

In plain words: an invariant subspace means a smaller space than the actual space V, where the application of any matrix in the representation does not leave the space. In terms of matrices, this means that there is an equivalent representation that can be written as a block matrix with a zero block:

$$D(g) = \begin{pmatrix} Q(g) & 0 \\ T(g) & P(g) \end{pmatrix} \tag{2.3}$$

Indeed with these notations, if we write all vector in V as $|x\rangle = \begin{pmatrix} v \\ w \end{pmatrix}$, we see that the subspace defined by vectors $|w\rangle = \begin{pmatrix} 0 \\ w \end{pmatrix}$ is transformed as

$$D(g)|w\rangle = \begin{pmatrix} 0 \\ P(g)w \end{pmatrix} \tag{2.4}$$

so that such vectors never leave the subspace.

If a representation is reducible, then there is a base such that all matrices can be written as such block matrices in the basis. A particular case of reducibility is *complete reducibility*, in which case $T(g) = 0$ as well.

Definition 2.2.3 — Completely Reducible representation. A representation $D(g)$ of a group G is completely reducible there exists a subspace W such that both W and its orthogonal W^T are both invariant.

In this case, we can thus *reduce* the representation to two different representations that act in different orthogonal spaces and write

$$D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \tag{2.5}$$

[1] In linear algebra, two $n \times n$ matrices A and B are called similar if there exists an invertible n-by-n matrix P such that $B = P^{-1}AP$.

We may wonder if all reducible transformations are completely Reducible. Sadly, this is not the case. Here is an example: the matrices

$$M(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{2.6}$$

are a representation of the group $\mathbb{R},+$. Indeed, $M(x)M(y) = M(x+y)$. However, we cannot diagonalize such matrices.

R The good news, however, is that in this lecture we will limit ourselves to *unitary representations*. Since unitary transformations keep the orthogonality, this means our matrices will always be orthogonal! In this case, $T(g)$ must be zero in equation 2.3.

Definition 2.2.4 — Irreducible representation. An irreducible representation is a representation that is not reducible.

Obviously, representations that live in dimension 1 are irreducible. The main use of group theory in quantum mechanics is to *reduce* representations into a set of irreducible ones.

We shall develop the tools that will help us doing that in the next lectures. More precisely, we are going to do two things: i) Study the consequences of having an irreductible representation, and ii) see how to get an irreductible representation.

2.3 Some useful theorems

Let us give some useful theorems, that we shall demonstrate later on.

Lemma 2.3.1 Burnside lemma : For a finite group of order h, there is only a finite number n of irreducible representations $a = 1 \dots n$ of dimension l_a, and

$$\sum_{a=1}^{n} l_a^2 = h \tag{2.7}$$

Lemma 2.3.2 Number of Irreducible rep. : For a finite group of order h, the number of irrep is equal to the number of conjugacy classes:

$$N_r = N_c \tag{2.8}$$

Theorem 2.3.3 — Representation of Abelian groups. All irreducible representations of Abelian groups are scalar.

From this, we deduce that an Abelian group of order h has h scalar representations irreducible. This last theorem is proven in next chapter from Shur's lemma, but since we already conclude that the number of conjugacy class if $N_c = H$ for Abelian groups, the theorem also follows from the two lemmas.

Let us give an example for Z_3, with its table:

$$\begin{array}{c|ccc} * & e & a & b \\ \hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array} \tag{2.9}$$

One irreducible representation is $e = 1, a = 1, b = 1$, this is the trivial one. Another possibility is to use the complex root of unity $e = 1, a = e^{2i\pi/3}, b = e^{4i\pi/3}$. In fact, one can proceed by using any of the complex roots for a and raising its power! The last representation is thus $e = 1, a = e^{4i\pi/3}, b = e^{8i\pi/3} = e^{i2\pi/3}$. In general for the cyclic group Z_n one can obtain all N representtaions by taking one of the $N-$root of unity for a and taking its power.

3. Schur Lemmas in Representation Theory

3.1 Schur Lemmas between irreducible representations

Schur's lemma discusses the link between irreducible representations, and in particular their link with an operator that commutes with all elements of the representation. This is what will be interesting for us.

Let $D_1(g)$ and $D_2(g)$ be two irreducible representations of a group G, each acting on different vector spaces $\mathscr{H}_1$ and $\mathscr{H}_2$:

$$D_1 : G \to GL(\mathscr{H}_1) \quad (3.1)$$

$$D_2 : G \to GL(\mathscr{H}_2) \quad (3.2)$$

Suppose there exists a linear transformation (a matrix) $A : \mathscr{H}_1 \to \mathscr{H}_2$ such that $AD_1(g) = D_2(g)A \forall g \in G$. The first Schur lemma (or off-diagonal Schur lemma) states that

Lemma 3.1.1 — Schur lemma 1:. If D_1 and D_2 are not equivalent, then $A = 0$.

The second Schur lemma (or diagonal Schur lemma) states that:

Lemma 3.1.2 — Schur lemma 2:. If $D_1 = D_2 = D$ (and thus $\mathscr{H}_1 = \mathscr{H}_2$), then $A = \lambda \mathbb{I}$, with $\lambda \in \mathbb{C}$.

In short, if there exists an operator A that commutes with all element of two *irreducible* representations then Schur lemmas gives a very strong limit to what A can be: either a trivial diagonal matrix (if the representations are equivalent) or a zero one (if they are not)! This will prove to be useful, as we shall see that it often happens that the Hamiltonian commutes with each element of some group (this is the symmetry invariance in quantum mechanics).

3.2 Proof of Schur's lemma

Let us proove Schur's lemma. We are going to need the defintion of "kernel" and "image" of an operator.

Definition 3.2.1 — Kernel of an operator. The kernel Ker$A$ of an operator $A : V_1 \to V_2$ is the set of vector $\vec{v}_1 \in V_1$ such that $Av_1 = 0$.

Definition 3.2.2 — Image of an operator. The image ImA of an operator $A : V_1 \to V_2$ is the set of vector $\vec{v}_2 \in V_2$ for which $\exists v_1 \in V_1$ such that $v_2 = Av_1$.

Theorem 3.2.1 — Rank-Nullity theorem. For any operator $A : V_1 \to V_2$, define $\text{Rank}(A) = \dim[\text{Im}(A)]$ and $\text{Nullity}(A) = \dim[\text{Ker}(A)]$, then $\dim[V_1] = \text{Nullity}(A) + \text{Rank}(A)$.

3.2.1 Proof of lemma 1

Proof. For all $g \in G$ we have:

- $\forall \vec{v}_1 \in \mathrm{Ker}A$ we have $A(D_1(g)\vec{v}_1) = D_2(g)A\vec{v}_1 = 0$. This means that the vector $D_1(g)\vec{v}_1$ is also in the kernel of A. In other words a vector in $W = \ker A$ stays in W upon transformation by $D_1(g), \forall g$: W is thus a stable sub-space of $D_1(g)$.
- From a similar reasoning, we can deduce that the image $W' = \mathrm{Im}A$ is also a stable subspace for $D_2(g)$. Indeed, this requires implies that if a vector can be written as $\vec{v}_2 = A\vec{v}_1$, then $D_2(g)\vec{v}_2$ can also be written as $A\vec{v}_1'$. This is the case since $D_2(g)\vec{v}_2 = D_2(g)A\vec{v}_1 = AD_1(g)\vec{v}_1 = A\vec{v}'$.

We thus conclude that$W = \mathrm{Ker}A$ is a stable subspace $D_1(g)$ and that $W' = A$ is a stable subspace of $D_2(g)$. However, by assumption, both representations are irreducible, so the only subspaces are either 0 or the entire space. We thus have either:

- $\mathrm{Ker}A = 0$, in which case the image is not empty, so that $\mathrm{im}A = V_2$. But this implies that the transformation A is invertible, but then $A^{-1}D_2(g)A = D_1(g)\forall g$, and D_2 and D_1 are equivalent, which contradicts the hypothesis.
- $\mathrm{Ker}A = V_1$, in which case $A = 0$ (and the image is empty:$\mathrm{im}A = 0$).

■

3.2.2 Proof of lemma 2

In this case, we have a map between either the same, or between equivalent representations. Additionally, $V_1 = V_2 = V$, and A is a square matrix. If the representation are equivalent, we can always rotate the space so that they are indeed identical.

Let us consider then that $D_1(g) = D_2(g) = D(g)\forall g$.

Proof. By the fundamental theorem of algebra, it exists an eigenvalue $\lambda \in \mathbb{C}$ such that $\det(A - \lambda\mathbb{I}) = 0$. Consider then the equation

$$(A - \lambda\mathbb{I})D(g) = D(g)(A - \lambda\mathbb{I})\,. \tag{3.3}$$

so that if $v \in \mathrm{Ker}(A - \lambda\mathbb{I})$ then $D(g)v$ also in $\mathrm{Ker}(A - \lambda\mathbb{I})$. $W = \mathrm{Ker}(A - \lambda\mathbb{I})$ is thus a stable subspace of transformation by $D(g)\forall g$. Given $D(g)$ is irreductible, either $W = 0$ or $W = V$. W cannot be zero, because at least the eigenvector of A corresponding to λ is in W! Therefore $W = V$.

We this have $\mathrm{Ker}(A - \lambda\mathbb{I}) = V$, so that $(A - \lambda\mathbb{I}) = 0$ and therefore $A = \lambda\mathbb{I}$. ■

3.3 Application to quantum mechanics

Let us illustrate how Schur's lemmas can be useful in the context of quantum mechanics. Typically, one considers the Hamiltonian H, and G a symmetry group that commutes with H. More precisely, if we have a representation of this symmetry group over the Hilbert space $\mathscr{H}$ on which the systems, we have $D(g) : \mathscr{H} \to \mathscr{H}$, and $[D(g), H] = 0 \forall g \in G$.

3.3.1 A typical example

Typically $\mathscr{H}$ would be an infinite dimensional space, that forms a basis (for instance the Fourier basis). In an infinite dimensional space, we expect that $D(g)$ is reducible. So, if we work hard, we can find a basis of the Hilbert space that reduces the representation, that is we can recompose the space as $\mathscr{H} = \mathscr{H}_\infty \oplus \mathscr{H}_\in \oplus \dots$ where all the $\mathscr{H}_\rangle$ are invariant over the group transformation. At this point, we thus have $\forall g \in G, D(g) = D_1(g) \oplus D_2(g) \oplus D_3(g) \dots$, or equivalently in matrix form:

$$D(g) = \begin{pmatrix} D_1(g) & 0 & 0 & \dots \\ 0 & D_2(g) & 0 & \dots \\ 0 & 0 & D_3(g) & \dots \\ \dots & & & \end{pmatrix} \tag{3.4}$$

In this basis, we write the Hamiltonian (which happens to be Hermitian) as

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \dots \\ H_{21} & H_{22} & H_{23} & \dots \\ H_{31} & H_{32} & H_{33} & \dots \\ \dots \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \dots \\ H_{12}^* & H_{22} & H_{23} & \dots \\ H_{13}^* & H_{23}^* & H_{33} & \dots \\ \dots \end{pmatrix} \tag{3.5}$$

Now, let us see what Schur's lemma tells us. If $[D(g), H] = 0 \forall g \in G$ then we can apply the Schur lemma between all blocks in this decomposition (see next section). We thus see that all diagonal blocks must thus be constant ones (from the second lemma), while the only off-diagonal blocks that are non-zero are those between equivalent representations (from the first lemma). This means that the Hamiltonian (where we have assumed that only D_1 and D_2 were equivalent) reads as:

$$H = \begin{pmatrix} \lambda_1 \mathbb{I} & H_{12} & 0 & 0 & \dots \\ H_{21} & \lambda_2 \mathbb{I} & 0 & \dots \\ 0 & 0 & \lambda_3 \mathbb{I} & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots \end{pmatrix} \tag{3.6}$$

This allows us to considerably simplify the Hamiltonian, without even doing quantum mechanics computations, just from the role of symmetry. In fact, if we use the same representation for all equivalent ones, even the H_{21} become diagonal.

3.3.2 Example for the parity group

A parity transformation (also called parity inversion) is the flip in the sign of a spatial coordinate. In three dimensions, it refers to the simultaneous flip in the sign of all three spatial coordinates (a point reflection): $\mathbf{P} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$.

A wave function can always be decomposed into an even and an odd component $\psi(\vec{x}) = \psi^+(\vec{x}) + \psi^-(\vec{x})$, and the application of the parity operator transforms it as

$$\mathbf{P}\psi(\vec{x}) = \mathbf{P}\psi^+(\vec{x}) + \mathbf{P}\psi^-(\vec{x}) = \psi^+(-\vec{x}) + \psi^-(-\vec{x}) = \psi^+(\vec{x}) - \psi^-(\vec{x}) \tag{3.7}$$

Note in particular that $\mathbf{PP} = 1$. The set of all parity transformations that can be obtained by the parity operator is thus limited to 2. The set of of these transformations forms the parity group $Z_2 = \{e, p\}$ that has the following Cayley table:

	e	p
e	e	p
p	p	e

In particular, this group has only two possible irreducible representations in dimension 1 on $\mathbb{R}$: (i) $D_1(g) = 1$ for $g = e, p$ and (ii) $D_2(e) = 1, D_2(p) = -1$.

Consider now a problem with a Hamiltonian that commutes with any parity transformation. The Hamiltonian lives in a large (possibly infinite) Hilbert space $\mathscr{H}$. Now, we consider a basis of $\mathscr{H}$ made of even and odd functions (such as the Fourrier basis): $\left\{\phi_1^+(x), \phi_2^+(x), \dots, \phi_1^-(x), \phi_2^-(x), \dots\right\}$.

This basis defines invariant subspaces with respect to parity. Indeed, we have the decomposition into irreducible representations of $D(e)$ and $D(p)$ as

$$D(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots \end{pmatrix} \text{ and } D(p) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ \dots \\ \dots & \dots & -1 & 0 & 0 \\ \dots & \dots & 0 & -1 & 0 \\ \dots \end{pmatrix}$$

Applying the Schur lemma, we now obtain that

$$H = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix} \tag{3.8}$$

3.4 Basis notation

When we reduce a representation into an irreducible one, we obtain a basis that is a direct sum of irreducible representations. We are going to denote the element in this basis using 3 indices as : $\{|a,j,x\rangle\}$. Here a will denote the type of representation, j the vector that forms the basis of this representtaion, and x the multiplicity of the representaion .

- $a = 1,2,3,\dots$ denote the number of the non-equivalent representations $D_1, D_2, D_3, \dots$. Here we have $D_j \neq D_k, \forall i \neq k$. At this point: $D_a(g)$ acts in a subspace $\mathscr{H}_a$, and in this subspace; the basis is given by $\{|a,j,x\rangle\}$ with $j = 1,2,3,\dots$ and $x = 1,2,3,\dots$.
- In this subspace; the basis is given by $\{|a,j,x\rangle\}$ with $j = 1,2,3,\dots$.
- Finally, the same representation can be used multiple times, as we have seen in the previous example. x is here to denote which of these equivalent representation we consider.

In other words, we denote the reduced representation as $D(g) = \oplus_{a,x} D_{a,x}(g)$, or equivalently

$$D(g) = \begin{pmatrix} D_{1,1}(g) & 0 & 0 & 0 & \dots \\ 0 & D_{1,2}(g) & 0 & 0 & \dots \\ 0 & 0 & D_{2,1}(g) & 0 & \dots \\ 0 & 0 & 0 & D_{2,2}(g) & \dots \\ \dots \end{pmatrix}$$

In this basis, that reduces the representation of the group G, we are interested by the element of an operator A as

$$\langle a,j,x| A |b,k,y\rangle \tag{3.9}$$

For instance we can write

$$\langle a,j,x| D(g) |b,k,y\rangle = \delta_{a,b}\delta_{x,y} \left[D_{a,x}(g)\right]_{jk} \tag{3.10}$$

Schur lemmas have direct consequences on eq.3.9. First, let us check that if $[A, D(g)] = 0, \forall g \in G$, this remains true for all the block in A acting on the different $D_{a,x}$. We write

$$\begin{aligned} 0 &= \langle a,j,x| [A,D(g)] |b,k,y\rangle = \langle a,j,x| (AD(g) - D(g)A) |b,k,y\rangle \\ &= \langle a,j,x| (A\mathbb{I}D(g) - D(g)\mathbb{I}A) |b,k,y\rangle \\ &= \sum_{b',k',y'} \left(\langle a,j,x| A |b',k',y'\rangle \langle b',k',y'| D(g) |b,k,y\rangle - \langle a,j,x| D(g) |b',k',y'\rangle \langle b',k',y'| A |b,k,y\rangle \right) \end{aligned}$$

Now, we shall use the delta in eq.3.10 so that

$$0 = \sum_{k'} \Big(\langle a,j,x| A |b,k',y\rangle \left[D_{b,y}(g)\right]_{k',k} - \left[D_{a,x}(g)\right]_{j,k'} \langle a,k',x| A |b,k,y\rangle \Big)$$

This equation tells us that indeed the block of A are satisfying the comutation relation, i.e. that for the representaion $D_{b,y}$ and $D_{a,x}$ we have $A_{(a,x),(b,y)}D_{b,y} = D_{a,x}A_{(a,x),(b,y)}$. Now, we can apply Schur lemma between all these blocks:

- If $a = b$, we can apply the diagonal Schur lemma and get $\langle a, j, x| A |a, k, y\rangle = \lambda_a(x,y)\delta_{jk}$.
- If $a \neq b$, we can apply the off-diagonal Schur lemma and get $\langle a, j, x| A |b, k', y\rangle = 0$.

We can write this more concisely, in a very useful form that will be, after all, the only thing worth remembering:

$$\langle a, j, x| A |b, k, y\rangle = \lambda_a(x,y)\delta_{ab}\delta_{jk} \tag{3.11}$$

3.5 Proof of the representation of Abelian groups

From Schur's lemma, we can deduce something very important:

Theorem 3.5.1 — Representation of Abelian groups. All irreducible representations of Abelian groups are scalar.

Proof. Let $D(g)$ be an irreducible representation of an abelian group G. Then we have, $\forall g, h \in G$, $D(g)D(h) = D(h*g) = D(g*h) = D(h)D(g)$. Since $D(h)$ commutes with all $D(g)$, then from the second Schur lemma, it must be a matrix $\mathbf{1}\lambda$, and $D(h) = \mathbf{1}\lambda(h)$ for all h. Since it is also irreducible, then $D(h) = \lambda(h)$. ■

4. Characters, Orthogonality, and other theorems

We shall establish in this chapter some important mathematical properties of irreps. This will turn out to be quite technical, but also very useful to find and identify them!

Most of it will be consecquences of Shurs lemma. Starting with :

4.1 The great orthogonality theorem

Theorem 4.1.1 — Great Orthogonality Theorem. Let D_a and D_b be two non-equivalent unitary representations of a finite group G of order N. n_a and n_b are the dimensions of the vector space for D_a and D_b. Then the great orthogonality theorem states that

$$\sum_{g\in G}\frac{n_a}{N}[D_a(g)]^*_{jk}[D_b(g)]_{lm}=\delta_{ab}\delta_{jl}\delta_{km} \tag{4.1}$$

We shall see that the great orthogonality theorem is a consequence of Schur's lemma. We can decompose it in two parts:

$$\sum_{g\in G}[D_a(g)]^*_{jk}[D_b(g)]_{lm} = 0, \forall a\neq b \tag{4.2}$$

$$\sum_{g\in G}[D_a(g)]^*_{jk}[D_a(g)]_{lm} = \frac{N}{n_a}\delta_{jl}\delta_{km} \tag{4.3}$$

and we can prove both parts as follows:

Proof. Consider any matrix X and the matrix M defined as

$$M=\sum_{g\in G}D_1(g^{-1})XD_2(g) \tag{4.4}$$

Then we have, for any $y\in G$

$$\begin{aligned}
MD_2(y) &= \sum_{g\in G}D_1(g^{-1})XD_2(g)D_2(y)\\
&= \sum_{g\in G}D_1(y)D_1(y^{-1})D_1(g^{-1})XD_2(g)D_2(y)\\
&= D_1(y)\sum_{g\in G}D_1(y^{-1})D_1(g^{-1})XD_2(g)D_2(y)\\
&= D_1(y)\sum_{g\in G}D_1(y^{-1}g^{-1})XD_2(gy)\\
&= D_1(y)\sum_{g\in G}D_1((gy)^{-1})XD_2(gy)\\
&= D_1(y)\sum_{h\in G}D_1(h^{-1})XD_2(h)=D_1(y)M
\end{aligned}$$

We can thus use Schur's lemmas on M. Since D_1 and D_2 are not equivalent we have $M = 0$ so that

$$\sum_{g\in G}\sum_{jl}[D_1(g^{-1})]_{kj}X_{jl}[D_2(g)]_{lm} = 0 \tag{4.5}$$

but $D_1(g^{-1})$ is $D_1(g)^\dagger$ so that

$$\begin{aligned}\sum_{g\in G}\sum_{jl}[D_1(g)]^\dagger_{kj}X_{jl}[D_2(g)]_{lm} &= 0 \\ \sum_{g\in G}\sum_{jl}[D_1(g)]^*_{jk}X_{jl}[D_2(g)]_{lm} &= 0\end{aligned} \tag{4.6}$$

Using $X_{jl} = 0$ except for one pair jl for which $X_{jl} = 1$ leads to eq.(4.2).

We now turn to eq.(4.3). If we construct the matrix M using the same representation, we get again $MD(x) = D(x)M$ and by the second Schur lemma:

$$\sum_{g\in G} D(g^{-1})XD(g) = c(X)\mathbb{I} \tag{4.7}$$

which, in full matrix notation, means

$$\sum_{g\in G}\sum_{jl}[D(g)]^*_{jk}X_{jl}[D(g)]_{lm} = c(X)\delta_{km}$$

We just need to compute the constant. Let us work on the diagonal, when $k = m$, and sum over k so that we have

$$\begin{aligned}\sum_{g\in G}\sum_{jlk}[D(g^{-1})]_{kj}X_{jl}[D(g)]_{lk} &= n_a c(X) \\ \sum_{g\in G}\sum_{jl}X_{jl}\sum_{k}[D(g^{-1})]_{kj}[D(g)]_{lk} &= n_a c(X) \\ \sum_{g\in G}\sum_{jl}X_{jl}[D(g)D(g^{-1})]_{lj} &= n_a c(X) \\ \sum_{g\in G}\sum_{jl}X_{jl}\mathbb{I}_{lj} &= n_a c(X) \\ \sum_{g\in G}\mathrm{Tr}X &= n_a c(X) \\ c(X) &= \frac{N}{n_A}\mathrm{Tr}X\end{aligned}$$

Using again $X_{jl} = 0$ except for one pair jl for which $X_{jl} = 1$ leads to eq.(4.3). ■

4.2 First part of Burnside's lemma

One can rewrite the Great Orthogonality Theorem as

$$\sum_{g\in G}\left(\sqrt{\frac{n_a}{N}}[D_a(g)]^*_{jk}\right)\left(\sqrt{\frac{n_b}{N}}[D_b(g)]_{lm}\right) = \delta_{ab}\delta_{jl}\delta_{km} \tag{4.8}$$

With this notation, we see that if we define a $N-$dimensional vector $|a,j,k\rangle$ such that each of their N-component reads

$$|a,j,k\rangle_g = \sqrt{\frac{n_a}{N}}[D_a(g)]_{kj} \tag{4.9}$$

then all these vectors are orthogonal, in the sense that

$$\langle a,j,k|b,l,m\rangle = \delta_{a,b}\delta_{j,l}\delta_{k,m}$$

Note that these vectors are living in a $N-$dimensional space $\mathbb{C}^N$.

How many of these vectors can there be? Well, one can choose the value of a between the N_r representations, and for each, one can choose i and j, so denoting n_a the dimension of the representtaion a, one can choose between $n_a \times n_a$. In total, there are therefore $\sum_a^{N_r} n_a^2$ such vectors.

Now, fondamentally, there cannot be more than N orthogonal vector in a N dimensional space, we see that we have:

$$\sum_{a=1}^{N_R} n_a^2 \leq N \tag{4.10}$$

This is the first part of Burnside lemma. Actually, Burnside lemma tells us for a finite group that this inequality is always verified and that

$$\sum_{a=1}^{N_R} n_a^2 = N \tag{4.11}$$

4.2.1 Remark on Burnside

This is a very important result. it tells us that the number of non-equivalent represention must satisifes a rule that involves their dimension: the sum over all nonequivalent representation of their square dimension equal the order of the group. This gives a very strong limit! For instance, this is telling that there are only a finite number of possible representations, and certainly less or equal to the order of the group.

If we have $N=5$, one possibility is that we have $n_a=1$, five times. Another possibility is that we have one representation of dimension 2 once ($n_a=2$), and one with $n_a=2$ once. There is only one finite group of order 5, the abelian group $Z_5 = e,a,b,c,d$ with group multiplication given by addition modulo five. In this case we thus have 5 non-equivalent representations of the group Z_5, and they are all in dimension 1.

The inequivalent irreducible matrix representations are shown in the following table: note that they are indeed all one-dimensional:

$g\backslash$ representation	D_1	D_2	D_3	D_4	D_5
e	1	1	1	1	1
a	1	$e^{2i\pi/5}$	$e^{4i\pi/5}$	$e^{6i\pi/5}$	$e^{8i\pi/5}$
b	1	$e^{4i\pi/5}$	$e^{8i\pi/5}$	$e^{2i\pi/5}$	$e^{6i\pi/5}$
c	1	$e^{6i\pi/5}$	$e^{2i\pi/5}$	$e^{8i\pi/5}$	$e^{4i\pi/5}$
d	1	$e^{8i\pi/5}$	$e^{6i\pi/5}$	$e^{4i\pi/5}$	$e^{2i\pi/5}$

More generaically; for a Z_N group, we find that

$$D_n(a^j) = e^{2i\pi j \frac{n}{N}} \tag{4.12}$$

We can aslo write what the orthogonality condition reads

$$\sum_{j=0\in N-1} \frac{1}{N} e^{-2i\pi j \frac{a}{N}} e^{2i\pi j \frac{b}{N}} = \delta_{ab} \tag{4.13}$$

where we recognize a Fourier relation!

4.3 Traces of a representation

Consider a representation $D(g)$ of all the elements in a group G. We shall denote the trace of all these matrices as

$$\chi_D(x) = \mathrm{Tr}(D(x)) \tag{4.14}$$

Theorem 4.3.1 — Classes & Traces. In a representation D, all the elements which are in the same conjugacy class have the same trace.

Proof. If there exists u such that $x = u^{-1}yu$ then

$$\begin{aligned} \mathrm{Tr}(D(x)) &= \mathrm{Tr}(D(u^{-1}yu)) = \mathrm{Tr}(D(u^{-1})D(y)D(u)) = \mathrm{Tr}(D(u)D(u^{-1})D(y)) = \mathrm{Tr}(D(e)D(y)) \\ &= \mathrm{Tr}(D(y)) \end{aligned} \tag{4.15}$$

■

4.4 Petit Orthoginality theorem (by collumn)

By taking the trace of the Grand Orthogonality theorem, we find

Theorem 4.4.1 — Petit Orthogonality Theorem.

$$\sum_{g\in G} \chi_a^*(g)\chi_b(g) = N\delta_{a,b} \tag{4.16}$$

Equivalently, one can write it summing over the number of equivalent N_c so that

$$\sum_{\mu=1\in N_c} n_\mu \chi_a^*(C_\mu)\chi_b(C_\mu) = N\delta_{a,b} \tag{4.17}$$

The first consequence is that we can do as before and interpret it as an orthogonality relation of N_r (the number of representation) vectors in a space of dimension N_c (the number of equivalent classes). Indeed, define these N_r N_c-dimension vectors:

$$|a\rangle_\mu = \sqrt{\frac{n_\mu}{N}}\chi_a(C_\mu) \tag{4.18}$$

Then these vectors are all orthogonal. Since the maximum numbers of orthogonal vectors is N_C, we have

$$N_R \leq N_C \tag{4.19}$$

The number of representation is smaller or equal to the nmber fo conjugaison classes. Again, we shall see that the equality is tight!

4.5 Characters of a representation

Definition 4.5.1 — Character. The set of all traces $\{\chi_D(g)\}$ is called the character of the representation D.

Clearly, two equivalent representations have the same Character. Indeed if $D_2(g) = SD_1(g)S^{-1}$, then using the cyclic property of the trace we have $\mathrm{Tr}D_2(g) = \mathrm{Tr}[SD_1(g)S^{-1}] = \mathrm{Tr}D_1(g)$. In fact this is a sufficient condition as well:

Theorem 4.5.1 — Characters of Irreps. Two irreps are equivalent if and only if they have the same character.

Proof. We already proved that the condition is necessary. To prove it is sufficient we reason by contradiction. Assume two irreps D_1 and D_2 are not equivalents but have the same character. Then using the petit Orthogonality theorem, we find that the sum of (modulus of) trace squared should be zero, which is impossible. ■

Using this approach, we can now compute Degeneracy numbers for representations, that is computing how many irreps a given reductible representation contains. We first write:

$$D(g) = \oplus_{a,x} D_{a,x}(g) = \oplus_a b_a D_a(g) \tag{4.20}$$

And the question is how to find b_a? Using the characters of each irreps, we know that:

$$\chi_D(g) = \sum_a b_a \chi_a(g) \tag{4.21}$$

$$\chi_D(C_\mu) = \sum_a b_a \chi_a(C_\mu) \tag{4.22}$$

Multiply by $n_\mu \chi_b^*(C_\mu)$, the number of element in class C_μ, and sum over classes

$$\sum_{\mu=1}^{N_c} n_\mu \chi_b^*(C_\mu)\chi_D(C_\mu) = \sum_{\mu=1}^{N_c} n_\mu \sum_a b_a \chi_b^*(C_\mu)\chi_a(C_\mu) \tag{4.23}$$

$$= \sum_a b_a \sum_{\mu=1}^{N_c} n_\mu \chi_b^*(C_\mu)\chi_a(C_\mu) = \sum_a b_a N \delta_{a,b} = N b_b \tag{4.24}$$

so that

$$b_a = \frac{1}{N} \sum_{\mu=1}^{N_c} n_\mu \chi_a^*(C_\mu)\chi_D(C_\mu)\,. \tag{4.25}$$

We thus now have a formula for each number of irrep contained in a given representation:

Theorem 4.5.2 — Computing Degeneracy. Assume a decompostion in irreps as

$$D(g) = \oplus_{a,x} D_{a,x}(g) = \oplus_a b_a D_a(g) \tag{4.26}$$

then we have

$$b_a = \frac{1}{N} \sum_\mu n_\mu \chi_a^*(C_\mu)\chi_D(C_\mu) \tag{4.27}$$

Another interesting concequence of the petit theorem is the following one:

Theorem 4.5.3 — Sufficient condition for irreps. A necessary and sufficient condition for a representation D to be an irrep is that

$$\sum_{\mu=1}^{N_c} n_\mu |\chi(C_\mu)|^2 = N \tag{4.28}$$

Proof. Using eq.4.22 and the petit orthogonality theorem, we find that

$$\sum_{\mu=1}^{N_c} n_\mu |\chi(C_\mu)|^2 = \sum_{i,j} b_i b_j \sum_{\mu=1}^{N_c} n_\mu \chi_i(C_\mu)^* \chi_j(C_\mu) = N \sum_{i,j} b_i b_j \delta_{ij} = N \sum_i b_i^2 \tag{4.29}$$

Being irreducible means having only one of the b_i=1, which prove the theorem. ■

4.6 Burnside Lemma and Regular Representation

We can now proove Burnside lemma. Consider the regular representation (which we introduced in the previous chapter) that is obtained using $N \times N$ matrices for a finite group of order N. Then we have a amazing fact: Any irreducible representation D of G appears in the regular representation $dim(D)$ times:

Theorem 4.6.1 — Regular representation decomposition. Consider the regular representation of a group. Then we have the following decomposition in irrep

$$D^r(g) = \oplus_{a,x} D_{a,x}(g) = \oplus_a d_a D_a(g) \tag{4.30}$$

where d_a is the dimension of the representation a.

Proof. We simply apply

$$b_a = \frac{1}{N} \sum_{\mu} n_\mu \chi_a^*(C_\mu) \chi^r(C_\mu) \tag{4.31}$$

and using the fact that for the regular representtaion all characters are zero except for the one corresponding to e, we find

$$b_a = \frac{1}{N} \chi_a^*(C_e) = \frac{d_a}{N} N = d_a \tag{4.32}$$

■

This finally allows to prove Burnside's lemma, by simply counting the dimensions:

Lemma 4.6.2 — Burnside lemma.

$$\sum_{i=1}^{N_r} d_i^2 = N \tag{4.33}$$

4.7 The magic formula for representation numbers

There is a final formula which is very convenient. It starts from the following remark:

Theorem 4.7.1 Let D be an irrep, then the sum of all matrices $D(x)$ for x in a conjugacy class is a multiple of unity.

$$M = \sum_{i \in C_\mu} D(x_i) = m \mathbb{I} \tag{4.34}$$

Proof. By definition of a conjugaison class we have for any $y \in G$

$$D(y^{-1}) M D(y) = \sum_{i \in C_\mu} D(y^{-1} x_i y) = M \tag{4.35}$$

using the reordering theorem (in a class). Thus, the matrix M commutes with all the representatioon, and the second Schur lemma tells us it is proportioonal to identity. ■

We consider this for a class μ and a representation a, the matrices (called Dirac Characters):

$$M_\mu^a = \sum_{k=1}^{n_\mu} D^{(a)}(x_k) = m_\mu^a \mathbb{I} \tag{4.36}$$

Taking the trace on both side, we find

$$n_\mu \chi^a(C_\mu) = d_a m_\mu^a \tag{4.37}$$

while if we take the product of two matrices we finds

$$M_\mu^a M_\nu^a = \sum_{kl \in C_\mu C_\nu} D^a(x_k) D^a(x_l) = \sum_{kl \in C_\mu C_\nu} D^a(x_k x_l) \tag{4.38}$$

so that

$$M_\mu^a M_\nu^a = \sum_\lambda n_{\mu\nu\lambda} M_\lambda^a \tag{4.39}$$

Using 4.36 we find

$$m_\mu^a m_\nu^a = \sum_\lambda n_{\mu\nu\lambda} m_\lambda^a \tag{4.40}$$

and together with 4.37 we reach the magic formula between characters:

Theorem 4.7.2

$$n_\mu n_\nu \chi^a(C_\mu) \chi^a(C_\nu) = d_a \sum_{\lambda=1}^{N_C} n_{\mu\nu\lambda} n_\lambda \chi_a(C_\lambda) \tag{4.41}$$

4.8 Collumn Orthoginality theorem

From the Grand Orthogonality theorem, we can also derive (demo not given, see Savona page 37) another orthoginality theorem:

Theorem 4.8.1 — Collumn Orthogonality Theorem.

$$\sum_{a=1 \in N_r} \chi_a^*(C_\mu) \chi_b(C_\nu) = \frac{N}{n_\mu} \delta_{\mu,\nu} \tag{4.42}$$

Using the same vector representation, we now find that

$$N_c \leq N_r \tag{4.43}$$

We have thus obtained the very important results:

$$N_r = N_c \tag{4.44}$$

R Together with Burnside lemma, this leads to the fact that Abelian group of order N have always N representation of dimension 1.

4.9 Character tables for finite group

All these tools sounds like a long list of absrtact equalities, but they are really useful, as we shall see in the series (petite classes).

For finite group, it is easy to find the characterers listed in table in the litterature (google is your friend!), listed as follows:

irrep\ class	$C_1(e)$	C_2	C_3	C_4	C_5
D_1	1	1	1	1	1
D_2	d_2	$\chi_2(C_2)$	$\chi_2(C_3)$	$\chi_2(C_4)$	$\chi_2(C_5)$
D_3	d_3	$\chi_3(C_2)$	$\chi_3(C_3)$	$\chi_3(C_4)$	$\chi_3(C_5)$
D_4	d_4	$\chi_4(C_2)$	$\chi_4(C_3)$	$\chi_4(C_4)$	$\chi_4(C_5)$
D_5	d_5	$\chi_5(C_2)$	$\chi_5(C_3)$	$\chi_5(C_4)$	$\chi_5(C_5)$

And using such tables, the decomposition into irreps is quite easy to do!

4.10 Projectors

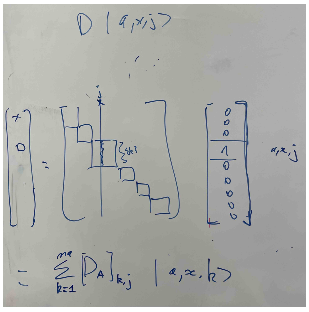
A last tool turns out to be quite useful in practice: The projectors! Let us consider the projectors on the basis $|a,j,x\rangle$.

To construct them, we start from

$$D(g) = \oplus_{a,x} D_{a,x}(g) \tag{4.45}$$

so applying the representation to these vectors we find

$$D(g)\,|a,j,x\rangle = \sum_k [D_a(g)]_{kj}\,|a,k,x\rangle \tag{4.46}$$



The result is not entirely intuite (but see image:)

Now we multiply by $[D_b(g)]^*_{k'j'}$ and sum over the group elements we find:

$$\sum_g [D_b(g)]^*_{k'j'} D(g)\,|a,j,x\rangle = \sum_k \sum_g [D_b(g)]^*_{k'j'} [D_a(g)]_{kj}\,|a,k,x\rangle\ , \tag{4.47}$$

and now, using the grand theorem of orthogonality, one finds

$$\sum_g [D_b(g)]^*_{k'j'} D(g)\,|a,j,x\rangle = \frac{N}{n_a}\delta_{ab}\delta_{jj'}\,|a,k',x\rangle \tag{4.48}$$

Thus, we see that we can define

$$\hat{\Pi}^b_{kj} = \frac{n_a}{N}\sum_g [D_b(g)]^*_{kj} D(g) \tag{4.49}$$

This operator satisfies

$$\hat{\Pi}^a_{kj}\,|a,j,x\rangle = |a,k,x\rangle \tag{4.50}$$

$$\hat{\Pi}^a_{kj}\,|b,j',x\rangle = 0\,\text{otherwise} \tag{4.51}$$

so that if we know one vector of the basis, then we can find all the other ones!

Now, let us take the trace. We tind

$$\hat{P}_a = \sum_j \hat{\Pi}^a_{jj} = \frac{n_a}{N}\sum_g \chi^*_a(g) D(g) \tag{4.52}$$

This is a projector on the basis of the representation! In other words we have

$$\hat{P}_a = \sum_{j,x} |a,j,x\rangle\,\langle a,j,x| \tag{4.53}$$

In summary

- P_a is a projector in the space generated by one irreducible representation, that is the space of all $|a,j,x\rangle$ for all j and x. That is, on the Hilbert space $\mathscr{H}_a = \oplus \mathscr{H}_{a,x}$.
- Π^a_{jj} is a projector on the subspace $|a,j,x\rangle$ for all x, but with a fixed j (that is , one one of the dimension of the representation).
- Π^a_{kj} is a generazlied projector.

5. Application on tensor products

5.1 Tensor products of matrices: the Kronecker product

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \tag{5.1}$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,2} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \\ a_{2,1} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{2,2} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}. \tag{5.2}$$

5.2 Tensorial product of representation

Let D_1 and D_2 be two representation of G on $\mathscr{H}_1$ and $\mathscr{H}_2$.

Let us consider now: $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$. Then $D(g) = D_1(g) \otimes D_2(g)$ is a representation of G on $\mathscr{H}$.

Easy to check:

$$\begin{aligned} D(g)D(f) &= [D_1(g) \otimes D_2(g)][D_1(f) \otimes D_2(f)] && (5.3) \\ &= D_1(g)D_1(f) \otimes D_2(g)D_2(f) && (5.4) \\ &= D_1(gf) \otimes D_2(gf) && (5.5) \\ &= D(gf) && (5.6) \end{aligned}$$

Good! So if we have a basis of $\mathscr{H}_1$ with a basis $|j_1\rangle, |j_2\rangle, \ldots$ and $\mathscr{H}_2$ with a basis $|k_1\rangle, |k_2\rangle, \ldots$ then we have a basis of $\mathscr{H}$ with a basis $|i_1\rangle \otimes |j_1\rangle, |i_2\rangle \otimes |j_2\rangle, \ldots$. Now we see in this basis that we have a representation of the group.

R The tensorial product of two irreducible representations DOES NOT have to be irreducible. In general, it is not.

Let us notice that if $D(g) = D_1(g) \otimes D_2(g)$, then the character is the product of the character:

$$\begin{aligned} \chi(g) &= \mathrm{Tr}(D(g)) && (5.7) \\ &= \mathrm{Tr}(D_1(g) \otimes D_2(g)) && (5.8) \\ &= \mathrm{Tr}(D_1(g))\mathrm{Tr}(D_2(g)) && (5.9) \\ &= \chi_1(g)\chi_2(g) && (5.10) \end{aligned}$$

5.3 Clebsch-Gordan decomposition

A tensor product of representation can be decomposed into a direct sum of irreducible representations. We have:

$$D^{(\alpha)} \otimes D^{(\beta)} = \oplus_\tau b_a D^{(a)} \tag{5.11}$$

This is often called a Clebsch-Gordan decomposition, and this formula is the Clebsch-Gordan series. The b denotes the degeneracy of a given decomposition. The change of basis from one basis of vector to the other defines the Clebsch-Gordan coefficients:

$$|a,j,x\rangle = \sum_{i,k} |\alpha,i\rangle \otimes |\beta,k\rangle \left(\langle\alpha,i| \otimes \langle\beta,k|\right) |a,j,x\rangle = \sum_{i,k} C^{aj}_{\alpha,i;\beta,k} |\alpha,i\rangle \otimes |\beta,k\rangle \tag{5.12}$$

Example: the well-known composition of cinetic moments. When we add two spins j_1 and j_2, we have a spin that can take values J from $j = |j_1 - j_2|$ to $j_1 + j_2$ (with unit degeneracy in this particular case):

$$D^{j_1} \otimes D^{j_2} = \oplus_{J=|j_1-j_2| \to j_1+j_2} D^J \tag{5.13}$$

and in the new basis, each representation J is dimension $2J+1$ (with $M = -J, -J+1, \dots, J$) and can be written as (here J plays the rolde of a and M the role of j, and x is not there because of the lack of degeneracy):

$$|J,M\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1,m_1;j_2,m_2\rangle\langle j_1,j_2;m_1,m_2 \mid J,M\rangle \tag{5.14}$$

$$= \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C^{J,M}_{j_1,m_1;j_2,m_2} |j_1,m_1;j_2,m_2\rangle \tag{5.15}$$

(5.16)

5.4 Example with C3v

Let us consider the symmetry of the triangle, the C3v group. It is a non abelian group of order 6, and its Cayley table reads:

$*$	e	c_3^1	c_3^2	σ	σ'	σ''
e	e	c_3^1	c_3^2	σ	σ'	σ''
c_3^1	c_3^1	c_3^2	e	σ'	σ''	σ
c_3^2	c_3^2	e	c_3^1	σ''	σ	σ'
σ	σ	σ''	σ'	e	c_3^2	c_3^1
σ'	σ'	σ	σ''	c_3^1	e	c_3^2
σ''	σ''	σ'	σ	c_3^2	c_3^1	e

(5.17)

The conjugacy classes are $C_e = \{e\}, C_1 = \left\{c_3^1, c_3^2\right\}$ and $C_2 = \{\sigma, \sigma', \sigma''\}$. Therefore there can be only 3 irreps. From Burnside lemma, we see that this imposes that there are two unidimensional representation, and one bidimensional one such that $1 + 1 + 2^2 = 6$.

Of course we have the trivial one where everyone is representation by a scalar equal to one. Since the value is equal to the trace in one dimension, all representation inside one call must be equal. With this in mind, we can find the second one quite easily: **One-dimensional representations:**

- A_1: $1,1,1,1,1,1$
- A_2: $1,1,1,-1,-1,-1$

Two-dimensional representation: We now needs to find the 2d representation. Since we are working with mirrors and rotation, this is quite easyly done in two dimension:

- Representation E: $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $c_3^1 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, $c_3^2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
- $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma' = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, $\sigma'' = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

We can now write the table of characters:

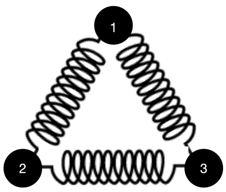
	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

Table 5.1: Character table for point group C3v

One can check that these are indeed irreducible representation since we can check that $\sum_{\mu=1}^{N_c} n_\mu |\chi(C_\mu)|^2 = N$

5.5 A practical case: a triangle of springs

Let us look of a physical system made of 3 beads attached by a triangle. The system is symmetric with respect to any transformation of C3v. Let us write the representaion of C3v in the 6-dimensional system that describe our problem physically. This can be actually done in two steps using a tensorial product.



First, we can see how each beads is transformed upon the action of the group. We write in a $3d$ space that

$$\vec{B}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{B}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{B}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.18}$$

Then we can see that the action of the group in this space is

- Representation $3d$: $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $c_3^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $c_3^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
- $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\sigma' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\sigma'' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Now if we look to the $6d$ representation in the system of coordinate $x_1, y_1, x_2, y_2, x_3, y_3$, we see that the effect of applying the transformation is, first, to exchange the balls, and then to physically perform either a rotation or a symmetry on the ball itself! In other words, the $6d$ representation can be written as

$$D^{6d} = D^{3d} \otimes D^{2d} \tag{5.19}$$

We finally can write the matrices of the $6d$ representation

- Representation $6d$: $e = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
- $c_3^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}$, $c_3^2 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$
- $\sigma = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$, $\sigma' = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$
- $\sigma'' = \begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Ok! This was long! Now, how can this 6d representation of $c3v$ can be decomposed? Let us compute the chractaer of this new representation! We find: $\chi_6(e) = 6, \chi_6(c_3^1) = \chi_6(c_3^2) = 0$ and $\chi_6(\sigma) = \chi_6(\sigma') = \chi_6(\sigma'') = 0$ as well. A moment of though reveal that we do not need to even apply our fancy formula! The only way this can be decomposed is

$$D^{3d}_{C3v} \otimes D^{2d}_{C3v} = D^{6d}_{C3v} = D_{A_1} + \oplus D_{A_2} + \oplus 2D_E \tag{5.20}$$

Let us now characterize what this means in the basis that perform this reduction. We shall use our projector formulas:

$$\hat{P}_a = \frac{n_a}{N} \sum_g \chi_a^*(g) D(g) \tag{5.21}$$

This is a projector on the basis of the representation! In other words we have

$$\hat{P}_a = \sum_{j,x} |a, j, x\rangle \langle a, j, x| \tag{5.22}$$

- First we look at the trivial representation. The projector reads

$$\hat{P}_{A_1} = \frac{n_a}{N}\sum_g \chi_a^*(g)D(g) = \frac{1}{6}\sum_g D(g) = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -\sqrt{3} & -1 & \sqrt{3} & -1 \\ 0 & -\sqrt{3} & \frac{3}{2} & \frac{\sqrt{3}}{2} & -\frac{3}{2} & \frac{\sqrt{3}}{2} \\ 0 & -1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \sqrt{3} & -\frac{3}{2} & -\frac{\sqrt{3}}{2} & \frac{3}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \tag{5.23}$$

This can be written as a rank one matrix, denoting the vector

$$|a\rangle = \left|0, \sqrt{\frac{1}{3}}, -\frac{1}{2}, -\sqrt{\frac{1}{12}}, \frac{1}{2}, -\sqrt{\frac{1}{12}}\right\rangle \tag{5.24}$$

we have

$$\hat{P}_{A_1} = |a\rangle\langle a| \tag{5.25}$$

This corresponding to a breathing mode, where everyone is moving together respecting the symmetry. In fact, we see that this mode respect entirely the c3v symmetry. This is always the case for the sector concerns by the trivial representaion: it transformes object that are completly symmetric to such transformation.

- Now we look at the second uni-dimensional representation. In this case we find

$$\hat{P}_{A_2} = \frac{n_b}{N}\sum_g \chi_b^*(g)D(g) = \frac{1}{6}\begin{pmatrix} 2 & 0 & -1 & \sqrt{3} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \sqrt{3} & 0 & -\frac{\sqrt{3}}{2} & \frac{3}{2} & -\frac{\sqrt{3}}{2} & -\frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & -\frac{3}{2} & \frac{\sqrt{3}}{2} & \frac{3}{2} \end{pmatrix} \tag{5.26}$$

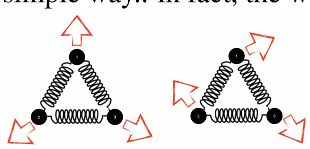
This can again be written as a rank one matrix, denoting the vector

$$|b\rangle = \left|\sqrt{\frac{1}{3}}, 0, -\sqrt{\frac{1}{12}}, \frac{1}{2}, -\sqrt{\frac{1}{12}}, -\frac{1}{2}\right\rangle \tag{5.27}$$

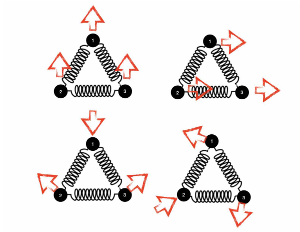
we have

$$\hat{P}_{A_2} = |b\rangle\langle b| \tag{5.28}$$

This correspond to a rotation mode! This is definitely breaking the symmetry, but in a rather simple way.. in fact, the way we break it depends on a single parameter (the angle) .



- Finally, we can do the same with the 2-degenerate 2d representation, and we find a rank 4 matrix that does not distinghish between the 2. Looking at the rank 4 matrix, we see that we there is abse where we have 2 translation modes (in x and y) and, additionally, some interesting vibration modes.



6. Symmetry and Wigner Theorem

In quantum physics, states are defined by kets denoted as $|\Psi\rangle$. However, we have some freedom: the same state can be defined by a *ray*, that is the set of vectors $e^{i\phi}|\Psi\rangle$. This "phase" freedom is important: it tells us that two observers that use two different vector that belong to the same ray should observe the same physics. This has important consequences in terms of symmetry.

6.1 Wigner theorem

Given two different observers with different kets that belong to the same ray, when we do a transformation, we need to have the same probability that is

$$P_1(|\psi\rangle_1 \to |\psi\rangle_1{}') = P_2(|\psi\rangle_2 \to |\psi\rangle_2{}') \tag{6.1}$$

here the $|\psi\rangle_1$ and $|\psi\rangle_2$ should be in the same ray, and $|\psi\rangle_1{}'$ and $|\psi\rangle_2{}'$ as well.

Theorem 6.1.1 — Wigner theorem. A symmetry/transformation operator can be either unitary and linear, or anti-unitary and anti-linear.

If it is unitary and linear we have:

- $U\alpha|\psi\rangle = \alpha U|\psi\rangle$
- $U(\alpha|\psi\rangle + \beta|\Phi\rangle) = \alpha U|\psi\rangle + \beta U|\Phi\rangle$
- $\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle$

This means, U is a unitary matrix ($U^T = U^{-1}$) that can be written as $U = e^{iT}$ and T is hermitian (and therefore can be an observable). Additionally, if the set of transformation forms a group, then we must have a set of U_α that is a *projective representation of the group*, that is a representation of the group up to phase.

$$U(f)U(g) = e^{i\xi(f,g)}U(fg) \tag{6.2}$$

If it is anti-unitary and anti-linear we have:

- $A\alpha|\psi\rangle = \alpha^* A|\psi\rangle$
- $A(\alpha|\psi\rangle + \beta|\Phi\rangle) = \alpha^* A|\psi\rangle + \beta^* A|\Phi\rangle$
- $\langle A\phi|A\psi\rangle = \langle\phi|\psi\rangle^* = \langle\psi|\phi\rangle$

We shall not give the proof in this course, but it can be found in Weinberg's Quantum theory of fields, page 50. Another reference is Michel Lebellac's book on quantum mechanics (in french), page 576.

Besides the fact that groups are thus useful in quantum mechanics, there shall be two additional fundamental consequences that we are going to explore:

1. We can use *projective* representations, and not only representations! This is very important. For instance, we shall see that $SU(2)$ will be the fundamental group for rotation in quantum mechanics, not $SO(3)$.
2. We are also allowed anti-unitary representaion. This is also useful —in a more limited but definitly real way— as it allows us to describe temporal inversion!

Wigner told us how to express such an operator A: We should just write:

$$A = UK \tag{6.3}$$

where U is a standard unitary operator, and K is the "complex conjugator" operator:

$$\begin{aligned} KzK &= z^* && (6.4)\\ K &= K^{-1} && (6.5)\\ K\langle\psi| &= \langle\psi^*| && (6.6) \end{aligned}$$

6.2 Temporal inversion operator

Let us define $\hat{T}$, the temporal inversion operator. Clearly its effect when applied to the kets $|r\rangle$ and $|p\rangle$ is (remember that speed is the derivative of position with respect to time):

$$\begin{aligned} \hat{T}|\vec{r}\rangle &= |\vec{r}\rangle && (6.7)\\ \hat{T}|\vec{p}\rangle &= |-\vec{p}\rangle && (6.8) \end{aligned}$$

We can also discuss how operators change when temporal inversion is applied. We claim:

$$\begin{aligned} \hat{T}\hat{\vec{R}}\hat{T}^{-1} &= \hat{\vec{R}} && (6.9)\\ \hat{T}\hat{\vec{P}}\hat{T}^{-1} &= -\hat{\vec{P}} && (6.10) \end{aligned}$$

Let us prove this claim:

$$\begin{aligned} \hat{T}\hat{\vec{P}}|\vec{p}\rangle &= \hat{T}\vec{p}\,|\vec{p}\rangle && (6.11)\\ &= \vec{p}\hat{T}\,|\vec{p}\rangle = \vec{p}\,|-\vec{p}\rangle = -(-\vec{p}\,|-\vec{p}\rangle) && (6.12)\\ &= -\hat{\vec{P}}|-\vec{p}\rangle && (6.13) \end{aligned}$$

but since

$$\hat{T}\hat{\vec{P}}|\vec{p}\rangle = \hat{T}\hat{\vec{P}}\hat{T}^{-1}\hat{T}\,|\vec{p}\rangle = \hat{T}\hat{\vec{P}}\hat{T}^{-1}\,|-\vec{p}\rangle \tag{6.14}$$

we reach the conclusion.

Finally, we cam investigate the effect on the operator $\hat{\vec{L}} = \hat{\vec{R}} \wedge \hat{\vec{P}}$ and show that

$$\hat{T}\hat{\vec{L}}\hat{T}^{-1} = -\hat{\vec{L}} \tag{6.15}$$

This is interesting because this means that spins (that are just intrinsic angular moments) should also transform this way!

6.2.1 $\hat{T}$ must be anti-unitary

Let us show now that we have no choice: the operator $\hat{T}$ must be anti unitary, and cannot be a unitary one.

Let us consider the time evolution of a given ket:

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle \tag{6.16}$$

If we apply the time-reversal operator, we find

$$\hat{T}\,|\psi(t)\rangle = |\psi(-t)\rangle = e^{iHt/\hbar}\hat{T}\,|\psi(0)\rangle \tag{6.17}$$

so we see here the anti-linearity effect! Additionally, since

$$\hat{T}\,|\psi(t)\rangle = \hat{T}e^{-iHt/\hbar}\,|\psi(0)\rangle \tag{6.18}$$

we have

$$\hat{T}e^{-iHt/\hbar} - e^{iHt/\hbar}\hat{T} = 0 \tag{6.19}$$

and for very small values of t we have

$$\hat{T}(1-iHt/\hbar) - (1+iHt/\hbar)\hat{T} = \frac{t}{\hbar}(\hat{T}H(-i) - iH\hat{T}) = 0 \tag{6.20}$$

If T is linear, we would have $\hat{T}H(-i) - H\hat{T} = -i(\hat{T}H + iH\hat{T}) = 0$ but we know that $\hat{H}$ commutes with the hamiltonian! So we have no choice but to follow Wigner, and accept its general form for an anti-unitary operator, it must read:

$$\hat{T} = \hat{U}K \tag{6.21}$$

6.2.2 Consequence on physics and representation of this operator?

What is the effect of such transformation on spin variables (e.g. Pauli matrices and cinetic moment)? Let us show some interesting relations:

$$T^2 = UKUK = UKUK^{-1} = UU^* \tag{6.22}$$

whre U^* is the complex conjugate (without transpose!!!). What is this ? We can see that

$$U^{-1} = U^\dagger = (U^*)^T \tag{6.23}$$

so we have

$$U^* = (U^{-1})^T \tag{6.24}$$

so that since twice reversing time should be like having not done anything (time reversal is just Z_2), and since we must have a projective representation, we must have

$$T^2 = U(U^{-1})^T = \eta\mathbb{I} \tag{6.25}$$

Let us find what η must be. We note that multiplying on the right by U^T

$$U(U^{-1})^T U^T = \eta U^T \tag{6.26}$$

$$U = \eta U^T \tag{6.27}$$

$$U^T = \eta U \tag{6.28}$$

so that

$$U = \eta U^T = \eta^2 U \tag{6.29}$$

so that $\eta = \pm 1$. This looks surprising, but after all, we are admitting such projective representations as possibilities according to Wigner's analysis.

Is the -1 actually happening sometime or are these just some mathematical irrelevant details? The answer is yes (and no to the second question), rather surprisingly. Consider the representations of this operator and its effect on Pauli matrices, and on angular moment operators in general.

We have:

$$S_z T\,|s,m\rangle = -TS_z\,|s,m\rangle = -m\hbar T\,|s,m\rangle \tag{6.30}$$

given the eigenket are not degenerate, we find that $T|s,m\rangle$ is an eigenket of S_z with eigenvalue $-m\hbar$, so that

$$T|s,m\rangle = c_m|s,-m\rangle \tag{6.31}$$

where c_m is a phase factor. Let us try to find the dependence. First, let us use instead the raising or lowering operators:

$$S_\pm = S_x \pm iS_y \tag{6.32}$$

we have

$$TS_\pm T^{-1} = TS_xT^{-1} \mp iTS_yT^{-1} = -S_x \pm iS_y = -S_\mp \tag{6.33}$$

if we now apply these operators, we find:

$$\begin{aligned}
S_+T|s,m\rangle &= -TS_-|s,m\rangle = -\hbar\sqrt{(s+m)(s-m+1)}T|s,m-1\rangle && (6.34)\\
&= -\hbar\sqrt{(s+m)(s-m+1)}c_{m-1}|s,-m+1\rangle && (6.35)\\
&= c_mS_+|s,-m\rangle && (6.36)\\
&= \hbar\sqrt{(s+m)(s-m+1)}c_m|s,-m+1\rangle && (6.37)
\end{aligned}$$

so that

$$-c_{m-1} = c_m \tag{6.38}$$

so that c_m changes its sign everytime m changes by a factor one. We can write equivalently:

$$T|s,m\rangle = e^{i\alpha}(-1)^{s-m}|s,-m\rangle \tag{6.39}$$

where α is a phase that has to be independent of m. Note that we have written the exponent of -1 as $s-m$ because if s is half-integer then so is m, but $s-m$ is always an integer.

At this point, we have

$$\begin{aligned}
T^2|s,m\rangle &= e^{-i\alpha}(-1)^{s-m}T|s,-m\rangle = (-1)^{s-m}(-1)^{s+m}|s,m\rangle && (6.40)\\
&= (-1)^{2s}|s,m\rangle && (6.41)
\end{aligned}$$

so at this point, we can conclude: if s is an integer, then $T^2 = \mathbb{I}$. If whoever s is a half-integer, that is in the fermionic case, we find the rather surprising/disturbing conclusion that $T^2 = -\mathbb{I}$! This property of "projective representation" seems to be real after all!

Can we deduce an interesting physical implication? The answer is yes. Consider the following equation:

$$\begin{aligned}
\langle T^2\psi|T\psi\rangle &= \langle T\psi|\psi\rangle^* = \langle\psi|T\psi\rangle && (6.42)\\
&= \pm\langle\psi|T\psi\rangle && (6.43)
\end{aligned}$$

If we had a "+", in the bosonic case, this is a triviality... but if however, we have a minus, as for fermions, this implies that

$$\langle\psi|T\psi\rangle = 0 \tag{6.44}$$

so that $|\psi\rangle$ and $T|\psi\rangle$ are linearly independent. Therefore, if the Hamiltonian commutes with T, and there is a degeneracy for all states: it must be that all states, even the ground state, have at least a two-degeneracy. This is called Kramers degeneracy:

Theorem 6.2.1 The Kramers degeneracy theorem states that for every energy eigenstate of a time-reversal symmetric system with half-integer total spin, there is at least one more eigenstate with the same energy. In other words, every energy level is at least doubly degenerate if it has half-integer spin. The law is named for the Dutch physicist H. A. Kramers.

For instance, the energy levels of a system with an odd total number of fermions (such as electrons, protons and neutrons) remain at least doubly degenerate in the presence of purely electric fields (i.e. no external magnetic fields). The hydrogen (H) atom contains one proton and one electron, so that the Kramers theorem does not apply. The lowest (hyperfine) energy level of H is nondegenerate. The deuterium (D) isotope on the other hand contains an extra neutron, so that the total number of fermions is three, and the theorem does apply. The ground state of D contains two hyperfine components, which are twofold and fourfold degenerate.

7. A bit on Lie groups

7.1 $SO(3)$

Amoung Lie groups, of particular interest are the following ones:

- $U(n)$ the group of unitary matrix in dimension n.
- $SU(n)$ the group of unitary matrix in dimension n with determinant one (S stands for special).
- $O(n)$ the group of orthogonal matrix in dimension n.
- $SO(n)$ the group of orthogonal matrix in dimension n with determinant one. This is the group of rotation matrices in dimension n.

Let us discuss in particular $SO(3)$, that is rotation in 3d. We have

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.1}$$

7.2 Infinitesimal generator and Lie Algebra

It is particularly interesting to look at small rotations. For instance, if θ is small, we see that the rotation around the z axis reads

$$\begin{bmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{1} - i\theta \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.2}$$

More generally, we see that if we perform an infinitesimal rotation arround each axis with angles $\theta_x, \theta_y, \theta_z$, we have

$$\vec{V}' = (\mathbf{1} - i\theta_x J_x)(\mathbf{1} - i\theta_y J_y)(\mathbf{1} - i\theta_y J_y)\vec{V} \approx \left(\mathbf{1} - i\vec{\theta}\cdot\vec{J}\right)\vec{V} \tag{7.3}$$

with

$$\vec{\theta} = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \tag{7.4}$$

and

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, J_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, J_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.5}$$

Since we can always decompose large rotation as a succesions of small ones, this means that we should be able to integrate over these infinitesimal moves, so that

$$R(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}} \tag{7.6}$$

In fact, this can be checked directly. Using, say a rotation in the z axis, and expanding the exponential leads back to the standards expression.

This is actually a generic phenomenon for Lie groups. Since they are differentiable, it is always possible to write their effect as the exponential of a matrix multiplied by the parameters. The matrices $\vec{J}$ are called the "infinitesimal generator" of the group.

Let us look at the commutation relation of these generators. We find:

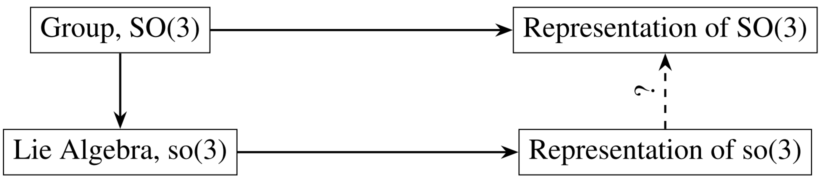
$$[J_i, J_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} J_k \tag{7.7}$$

with the Levi-Cevita symbol:

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) = (1,2,3), (2,3,1), \text{or } (3,1,2) \\ -1, & \text{if } (i,j,k) = (3,2,1), (1,3,2), \text{or } (2,1,3) \\ 0, & \text{otherwise} \end{cases} \tag{7.8}$$

The commutation relation is called the Lie Algebra of the group, and it is denoted as $so(3)$ (note the capital letters for the group, and the lower case for the Lie Algebra).

More precisely (and informally) A Lie algebra is a vector space where the elements are satisfying a particular commutation relation. In this sense, the matrices J_x, J_y, J_z are a *representation of the Lie Algebra* of $so(3)$: they define a particular representation, in dimension 3, of matrices that satifies the commutation relation of $so(3)$.



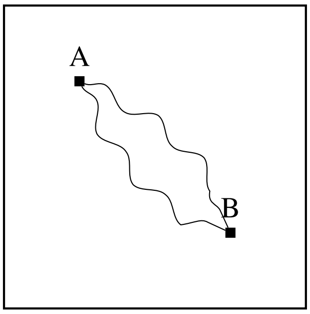
An important question now arise: We see that we have one representation of the Lie Algebra (the J) and if we exponentiate it, we find one representation of the Group (the R). However, we know that there are many representation of $SO(3)$ (in many dimension actyally), and we can guess that they are many representation of the Lie Algebra (after all, this is a VERY common commutation relation: all spins and cinetic moment operator will satisfie it!"

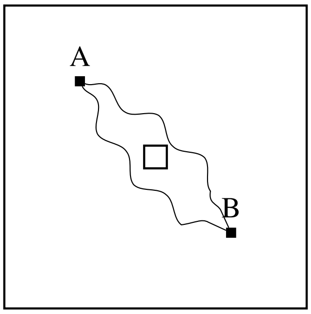
So is it true that ANY representation of the Lie Algebra will lead, upon exponentiation, to a representation of the group?

The answer to this question is, unfortunatly, **NO, in general!**, but **YES if the group is simply connected**. Topology plays an important role here: it is only for simply connected groups that any representation of the Lie Algebra is also a group representation.

7.3 Simply connected groups

Why is it so important that the topology of the group is simply connected? This has to do with analytic continuation and the fact that one can go from a succession of infinitesimal moves to a large one in a single, well-defined, way. This is only possible for Simply connected topologies (See below):





7.4 SU(2) vs SO(3)

7.4.1 SU(2)

Let us now look at SU(2). This is the group of matrices that depends on two complex numbers α and β as follows:

$$U(\alpha,\beta) = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \tag{7.9}$$

with the constraint that $\alpha\overline{\alpha}+\beta\overline{\beta} = 1$ (so that the determinant is indeed unity).

Using $\alpha = x+iy$ and $\beta = z+iw$, with $x = \sqrt{1-y^2-z^2-w^2}$, and taking the differential, one finds

$$dU(\alpha,\beta) = i\begin{bmatrix} dy & idz+dw \\ -idz+dw & -dy \end{bmatrix} - \frac{ydy+zdz+wdw}{\sqrt{1-y^2-z^2-w^2}}\mathbf{1} \tag{7.10}$$

so that the infinitesimal generators read, at the point $x = 1, y, w, w = 0$:

$$\frac{dU}{dy} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \frac{dU}{dz} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \frac{dU}{dw} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{7.11}$$

Up to some constant, we recognize that a possible base is the one of Pauli matrices, so that the infinitesimal generators reads:

$$U = e^{-i\vec{\theta}\cdot\vec{\sigma}} \tag{7.12}$$

with

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7.13}$$

With the commutation relation being the same as those of J_x, J_y and J_z! Therefore, we see that the $su(2)$ Lie algebra is isomorphic to the $so(3)$ Lie algebra! These groups have a deep relationship!

7.4.2 SO(3) is a representation of SU(2)

Let us investigate this relation by proving that SO(2) is a representation of SU(2) (but not the inverse).

We start with the following representation. For any point in $\vec{r}in\mathbb{R}^3$, we associate the following $d = 2$ Hermitian and zero trace matrix.

$$M(\vec{r}) = \begin{bmatrix} z & x+iy \\ x-iy & -z \end{bmatrix} \tag{7.14}$$

Now, let us look how SU(2) transform these matrices. We define the transformation as

$$M' = f_U(M) = UMU^{-1} \tag{7.15}$$

Clearly, such transformation respects the SU(2) structure. Indeed,

$$f_U(f_V(M)) = U(VMV^{-1})U^{-1} = UVM(UV)^{-1} = f_{UV}(M) \tag{7.16}$$

Additionally, these transformations keep the trace zero and preserve the hermitian property, so

$$M' = \begin{bmatrix} z' & x'+iy' \\ x'-iy' & -z' \end{bmatrix} \tag{7.17}$$

This means that our transformation on M implies, implicitly, a transformation in $3d$ with $\vec{r}' = f_U^{3d}(\vec{r})$. This transformation is continuous and linear Additionally, it also preserves the determinaent of M, and thus it preserve the lenght of the 3d vector $\vec{r}$. In this case it can only be rotation and/or mirrors. But mirrors are NOT continuous, and thus this means that $f_U^{3d}(\vec{r})$ is a rotation of the point $\vec{r}$.

We have thus created an interesting bridge: $f_U^{3d}(\vec{r})$ is just implementing rotations so that

$$f_U^{3d}(\vec{r}) = R(U)\vec{r} \tag{7.18}$$

Therefore, we have made (implicitly) a map between the SU(2) group to the SO(3) group: the matrix R(U) (which are just the rotation matrix of SU(3)) are a representation of SU(2), since they follow the $f_U^{3d}(\vec{r})$ transformation, and thus the SU(2) composition rule.

In other words SO(3) is a representation of SU(2).

However, the opposite is not true! In fact, it is easy to see that this relation is not an isomorphism, because U and $-U$ have the exact same effect in eq.(7.15). Therefore, both of them have the same representation in $SO(3)$: there are two elements of SU(2) for one element of SO(3). Notice, however, that these elements differs only by a minus sign and therefore:

SU(2) is not a representation of SO(3), but it is a projective representation of SO(3).

This has amazing and important concequences, because Wigner tells us that we are allowed projecrtive representation in quantuym mechanics. So it is possible that some object will transform with $SU(2)$ upon rotation! We know these object: half integer spins!

The fact that some mathematical objects transforms with SU(2) rather than SO(3) is a deep concequence of the law of quantum mechanics, and of Wigner theorem.

7.4.3 Topologies of SO(3) and SU(2)

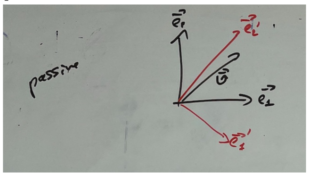
TBD

8. Continous transformation

8.1 Change of variables

Passive (change of basis) versus active (change of vector) point of view

passive



In the passive view, vectors are KEEPT, but axis are changed. We thus have

$$\vec{e}' = P\vec{e} \tag{8.1}$$

so that for a fixed vector $\vec{V}$, while the vector stay the same, we see that coordinate in the new basis are different and change as:

$$\vec{V}' = \sum_i v_i \vec{e}_i = \sum_i v_i P^{-1} \vec{e}_i' = \sum_i v_i' \vec{e}_i' \tag{8.2}$$

so, using matrix notations, the coordinate changes as

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = P^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{8.3}$$

Similarly, operator do not change, their components do! We have

$$O_{ij} = \vec{e}_i^{\,T} O \vec{e}_j \tag{8.4}$$

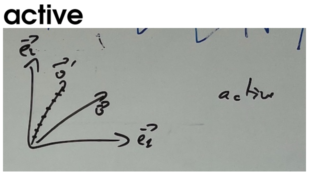
and in the new system of coordinate, we have

$$O'_{ij} = \vec{e}_i'^{\,T} O \vec{e}_j{}' = \vec{e}_i^{\,T} P^{-1} O P \vec{e}_i \tag{8.5}$$

So in the new basis, the expression of the operator, in the new coordinates, changes as

$$O' = P^{-1} O P \tag{8.6}$$

active



In the active view, the basis STAYS but the vectors are transformed (we change the systems! We have new vectors) Here the vectors are transformed so:

$$\vec{V}' = R\vec{V} \tag{8.7}$$

In this case, we can ask as well how is an operator transformed? Well, if we need another operator, then applied on the new vectors, it must gives the same results as the old operator applied on the old vectors, so

$$O'_{ij} = \vec{e}_i'^T O' \vec{e}_j{}' = \vec{e}_i{}^T O \vec{e}_i = O_{ij} \tag{8.8}$$

This means that in this view, we must have

$$\vec{e}_i'^T O' \vec{e}_j{}' = \vec{e}_i{}^T R^T O' R \vec{e}_j = \vec{e}_i{}^T O \vec{e}_i \tag{8.9}$$

so that

$$O' = ROR^{-1} \tag{8.10}$$

We should be careful as to which point of view we use, since we see they leads to exactly opposite transformations!

8.2 Rodriguez formula

Let us adopt the active point of view, with trigonometric convention. If $\vec{v}$ is a vector in $\mathbb{R}^3$ and $\vec{n}$ is a unit vector describing an axis of rotation about which $\vec{v}$ rotates by an angle θ the Rodrigues formula for the rotated vector reads :

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v}\cos\theta + (\mathbf{n}\times\mathbf{v})\sin\theta + \mathbf{n}\,(\mathbf{n}\cdot\mathbf{v})(1-\cos\theta)\,. \tag{8.11}$$

It is easily checked that, indeed, for rotation about the x, y and z axis, it gives back the usual rotation matrices:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.12}$$

As we have seen, this can also be written in the exponential form. We are going to change a bit notation to be more "quantum" by adding a planck constant in there! From now on we have:

$$R(\vec{\theta}) = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{J}} \tag{8.13}$$

with

$$J_x = \hbar \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, J_y = \hbar \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, J_z = \hbar \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.14}$$

with commutation

$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk} J_k \tag{8.15}$$

8.3 How do Wave functions transforms

Active transformation: Say that we transform the system activaly and write

$$\vec{r}' = R\vec{r} \tag{8.16}$$

then the new wave function at the new point must be the same as the old function in the old point. Therefore:

$$\Phi'(\vec{r}') = \Phi(\vec{r}) = \Phi(R^{-1}\vec{r}') \tag{8.17}$$

In a nutshell the function transforms as :

$$\Phi'(\vec{r}) = \Phi(R^{-1}\vec{r}) \tag{8.18}$$

or in terms of operators

$$\Phi'(\vec{r}) = \mathscr{R}\Phi(\vec{r}) = \Phi(R^{-1}\vec{r}) \tag{8.19}$$

We thus have an operator that transforms wave functions that must act as

$$\langle\vec{r}|\mathscr{R}|\Phi\rangle = \left\langle R^{-1}\vec{r}\middle|\Phi\right\rangle \tag{8.20}$$

Let us look at a concrete example beyond rotations and look at translations. We have

$$\hat{T}_{\vec{a}}\phi(\vec{x}) = \phi(\vec{x}-\vec{a}) \tag{8.21}$$

In terms of infinitesimal changes, this yields (in one dimension)

$$\begin{aligned} \hat{T}_{\varepsilon}\phi(x) &= \phi(x-\varepsilon) && (8.22)\\ &\approx \phi(x) - \varepsilon\phi'(x) \approx (1-\varepsilon\frac{\partial}{\partial x})\phi(x) && (8.23)\\ &\approx e^{-\frac{i\varepsilon}{\hbar}(-i\hbar\frac{\partial}{\partial x})}\phi(x) \equiv e^{-\frac{i\varepsilon}{\hbar}\hat{P}}\phi(x) && (8.24) \end{aligned}$$

As a consequence, we see that the translation operator is simply the exponential of the momentum operator!

$$\hat{T}_{\vec{a}} = e^{-\frac{i}{\hbar}\vec{a}\cdot\hat{\vec{P}}} \tag{8.25}$$

This is one example of a transformation operator being unitary, and thus being an exponential of a Hermitian operator. Additionally, this illustrates Noether theorem: symmetry with respect to translation implies the conservation of momentum.

8.4 Generic transformation

In general, for a ket $|\psi\rangle \in \mathscr{H}$ we must have something as

$$|\psi\rangle' = \mathscr{R}|\psi\rangle \tag{8.26}$$

with $\mathscr{R}$ a (projective) representation of the group of transformations so that

$$\mathscr{R} = \hat{D}(R) \tag{8.27}$$

We shall define a "vector" in the physics sense as quantity that transforms upon rotation with the $SO(3)$ matrix. Not ANY representation, but INDEED the $3d$ one of SO(3) matrix!

From this point of view, the physical ket is NOT a vector! There is an important distinction between mathematicians and physicists: for a mathematician a vector is an element of a vector space, but this is too trivial for physicists! Again We shall call vector objects that transforms with $SO(3)$ upon rotation!

What is an example of a vector? Well, position is a vector, and impulsion as well.

That being said, we know that D must be reducible in a given base, so we should look for one and work in it! A good basis is the "spherical basis" where we have representation of the form

$|j,m\rangle$. This basis has a nice expression in angular representation since $\langle\theta,l|j,m\rangle = Y_l^m(\theta,l)$, where $Y_l^m(\theta,l)$ are the spherical harmonics!

In this basis we can write (as we have seen) all representations as

$$D^j(R) = e^{-i\theta\frac{\vec{n}\vec{J}}{\hbar}} \tag{8.28}$$

where the Js are any of the matrix satisfying the SU(2) lie algebra. We know these matrix quite well actually since these are the usual spin matrices!

The representation are often called "Wigner D-matrix" in the spherical basis! in full generality we write the Wigner D-matrix as a square matrix of dimension $2j+1\times 2j+1$ as

$$\mathscr{D}^j_{m'm}(\alpha,\beta,\gamma) \equiv \langle jm'|D(\alpha,\beta,\gamma)|jm\rangle = e^{-im'\alpha}d^j_{m'm}(\beta)e^{-im\gamma}. \tag{8.29}$$